

微积分第一次作业 ($W \times f$)

2022.2.25

1

设 $\|\cdot\|$ 是范数, 证明 $\mathbb{C} = \{\vec{x} \in \mathbb{R}^m \mid \|\vec{x}\| \leq 1\}$ 是闭凸集。

$$(\mathbb{C} \in \mathbb{R}^m \text{ 是凸集} \iff \forall \vec{x}, \vec{y} \in \mathbb{C}, \forall t \in (0, 1), (1-t)\vec{x} + t\vec{y} \in \mathbb{C})$$

证明: 先证明其为凸集, 由范数的性质, 满足三角不等式和正齐次性有

$$\|\vec{x}\|, \|\vec{y}\| \leq 1 \in \mathbb{C}, \forall t \in (0, 1), \|(1-t)\vec{x} + t\vec{y}\| \leq (1-t)\|\vec{x}\| + t\|\vec{y}\| \leq 1$$

考虑 $\mathbb{C}$ 的补集 $\mathbb{A} = \{\vec{x} \in \mathbb{R}^m \mid \|\vec{x}\| > 1\}$, 对于该集合中任意一点 $\vec{a} \in \mathbb{A}$, 令 $\|\vec{a}\| = \lambda > 1$

$$\forall \vec{a} \in \mathbb{A}, \exists \delta = \frac{\lambda - 1}{2}, \text{ s.t. } \forall \|\vec{x} - \vec{a}\| < \delta, \|\vec{x}\| \geq \|\vec{a}\| - \|\vec{a} - \vec{x}\| = \|\vec{a}\| - \|\vec{x} - \vec{a}\| > 1$$

从而 $\mathbb{A}$ 满足任意一点总存在该点领域全在 $\mathbb{A}$ 中, $\mathbb{A}$ 为开集, $\mathbb{C}$ 为 $\mathbb{A}$ 的补集, 则 $\mathbb{C}$ 为闭集

再结合其为凸集, 得 $\mathbb{C}$ 为闭凸集 $\square$

2

设 $\|\cdot\|$ 满足正定、正齐次性, 证明: $\|\cdot\|$ 是范数 $\iff \mathbb{C} = \{\vec{x} \in \mathbb{R}^m \mid \|\vec{x}\| \leq 1\}$ 是凸集。

证明: 由第一题结论可得, $\|\cdot\|$ 是范数 $\implies \mathbb{C} = \{\vec{x} \in \mathbb{R}^m \mid \|\vec{x}\| \leq 1\}$ 是凸集。

又若 $\mathbb{C} = \{\vec{x} \in \mathbb{R}^m \mid \|\vec{x}\| \leq 1\}$ 是凸集, 即 $\forall \vec{x}, \vec{y} \in \mathbb{C}, \forall t \in (0, 1), (1-t)\vec{x} + t\vec{y} \in \mathbb{C}$ 。若

$\|\vec{x}\|$ 或 $\|\vec{y}\|$ 中有一者为 0, 由 $\|\cdot\|$ 的正定性, 可得 $\vec{x}$ 或 $\vec{y}$ 中一者为 $\mathbf{0} \in \mathbb{R}^m$, 此时三角不等式

$\|\vec{x}\| + \|\vec{y}\| \geq \|\vec{x} + \vec{y}\|$ 成立。则对两者范数均大于 0 的情况, $\forall \vec{x}, \vec{y}$, 令 $\|\vec{x}\| = l_1 > 0$

$\|\vec{y}\| = l_2 > 0$, 单位化有 $\vec{n}_1 = \frac{\vec{x}}{l_1}, \vec{n}_2 = \frac{\vec{y}}{l_2}$, 满足 $\vec{n}_1, \vec{n}_2 \in \mathbb{C}$, 由凸集性质

$$\begin{aligned} \because \forall t \in (0, 1), \|(1-t)\vec{n}_1 + t\vec{n}_2\| \in \mathbb{C}, \vec{n}_1 &= \frac{\vec{x}}{l_1}, \vec{n}_2 = \frac{\vec{y}}{l_2} \\ \therefore \forall t \in (0, 1), \|\frac{1-t}{l_1} \cdot \vec{x} + \frac{t}{l_2} \cdot \vec{y}\| \leq 1 \therefore \text{set } t &= \frac{l_2}{l_1 + l_2}, \|\frac{\vec{x} + \vec{y}}{l_1 + l_2}\| \leq 1 \end{aligned}$$

由正齐次性, $\|\frac{\vec{x} + \vec{y}}{l_1 + l_2}\| = \frac{1}{l_1 + l_2} \|\vec{x} + \vec{y}\| \leq 1$, 则有 $\|\vec{x} + \vec{y}\| \leq l_1 + l_2 = \|\vec{x}\| + \|\vec{y}\|$

则 $\mathbb{C} = \{\vec{x} \in \mathbb{R}^m \mid \|\vec{x}\| \leq 1\}$ 是凸集 $\implies \|\cdot\|$ 为范数, 两者等价 $\square$

3

证明 $\|\vec{x}\|_\infty = \max_{1 \leq i \leq m} |x^i|$ 是范数。

证明：显然正定性满足；且 $\|\lambda \vec{x}\|_\infty = \max_{1 \leq i \leq m} |\lambda x^i| = |\lambda| \max_{1 \leq i \leq m} |x^i| = |\lambda| \|\vec{x}\|_\infty$ 满足正齐次性，

有 $\vec{x} = (x^1, \dots, x^m)^T, \vec{y} = (y^1, \dots, y^m)^T$ ，设两者最大值为 $|x^{max}|, |y^{max}|$

$$\begin{aligned} \|\vec{x} + \vec{y}\|_\infty &= (x^1 + y^1, x^2 + y^2, \dots, x^m + y^m) \leq (|x^1| + |y^1|, \dots, |x^m| + |y^m|) \\ &\leq (|x^{max}| + |y^1|, \dots, |x^{max}| + |y^m|) = |x^{max}| + (|y^1|, \dots, |y^m|) \leq |x^{max}| + |y^{max}| \end{aligned}$$

即有 $\forall \vec{x}, \vec{y}, \|\vec{x} + \vec{y}\|_\infty \leq \|\vec{x}\|_\infty + \|\vec{y}\|_\infty$ ，满足三角不等式，从而 $\|\vec{x}\|_\infty$ 是（无穷）范数 $\square$

4

设 $p > 1$ ，证明 $\|\vec{x}\|_p = (\sum_{i=1}^m |x^i|^p)^{\frac{1}{p}}$ 是范数。

证明：显然 $(\sum_{i=1}^m |x^i|^p)^{\frac{1}{p}} \geq 0$ 满足正定性，若乘以 λ 倍，有

$$\|\lambda \vec{x}\|_p = \|\vec{x}\|_\infty = (\sum_{i=1}^m |\lambda x^i|^p)^{\frac{1}{p}} = \|\vec{x}\|_\infty = (\lambda^p \sum_{i=1}^m |x^i|^p)^{\frac{1}{p}} = \lambda \cdot (\sum_{i=1}^m |x^i|^p)^{\frac{1}{p}} = \lambda \|\vec{x}\|_\infty$$

从而 $\|\vec{x}\|_p$ 满足正齐次性，对三角不等式，先证明两个不等式

引理一：Young 不等式：设 $a, b > 0, q, p > 1, \frac{1}{p} + \frac{1}{q} = 1$ ，有以下不等式成立

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

构造函数 $f(x) = \frac{1}{p}x^p + \frac{1}{q} - x$ ($x > 0$)，求导有 $f'(x) = x^{p-1} - 1$ ，当 $x \in (0, 1)$ 时，

$f'(x) < 0$ ， $f(x)$ 单调递减；当 $x \in (1, +\infty)$ 时， $f'(x) > 0$ ， $f(x)$ 单调递增。从而

$f(x) \geq f(1) = \frac{1}{p} + \frac{1}{q} - 1 = 0$ 。令 $x = \frac{a}{b^{\frac{q}{p}}}$ 有 $\frac{1}{p} \frac{a^p}{b^q} + \frac{1}{q} \geq \frac{a}{b^{\frac{q}{p}}}$ ，两边乘以 b^q ，代入

$$\frac{1}{p} = 1 - \frac{1}{q}，则有 \frac{a^p}{p} + \frac{b^q}{q} \geq a \cdot b^{q - \frac{q}{p}} = ab$$

引理二：Hölder 不等式：设 $a_k, b_k > 0, p, q \geq 1, k = 1, 2, \dots, n, \frac{1}{p} + \frac{1}{q} = 1$ ，有以下不等式成立

$$\sum_{k=1}^n a_k b_k \leq (\sum_{k=1}^n a_k^p)^{\frac{1}{p}} \cdot (\sum_{k=1}^n b_k^q)^{\frac{1}{q}}$$

考虑两者相除运用 Young 不等式即可得证

$$\begin{aligned} \frac{\sum_{k=1}^n a_k b_k}{\left(\sum_{k=1}^n a_k^p\right)^{\frac{1}{p}} \cdot \left(\sum_{k=1}^n b_k^q\right)^{\frac{1}{q}}} &= \sum_{k=1}^n \left(\left(\frac{a_k^p}{\sum_{k=1}^n a_k^p}\right)^{\frac{1}{p}} \cdot \left(\frac{b_k^q}{\sum_{k=1}^n b_k^q}\right)^{\frac{1}{q}} \right) \\ &\leq \frac{1}{p} \sum_{k=1}^n \frac{a_k^p}{\sum_{k=1}^n a_k^p} + \frac{1}{q} \sum_{k=1}^n \frac{b_k^q}{\sum_{k=1}^n b_k^q} = \frac{1}{p} + \frac{1}{q} = 1 \end{aligned}$$

拆开和的 p -范数有 $\sum_{i=1}^n (x^i + y^i)^p = \sum_{i=1}^n x^i (x^i + y^i)^{p-1} + \sum_{i=1}^n y^i (x^i + y^i)^{p-1}$

令 $\frac{1}{p} + \frac{1}{q} = 1$ ，利用 Hölder 不等式放缩两者有

$$\begin{aligned} \sum_{i=1}^n x^i (x^i + y^i)^{p-1} &\leq \left(\sum_{i=1}^n (x^i)^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^n (x^i + y^i)^{(p-1)q}\right)^{\frac{1}{q}} = \left(\sum_{i=1}^n (x^i)^p\right)^{\frac{1}{p}} \cdot \left(\sum_{i=1}^n (x^i + y^i)^p\right)^{\frac{1}{q}} \\ \sum_{i=1}^n y^i (x^i + y^i)^{p-1} &\leq \left(\sum_{i=1}^n (y^i)^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^n (x^i + y^i)^{(p-1)q}\right)^{\frac{1}{q}} = \left(\sum_{i=1}^n (y^i)^p\right)^{\frac{1}{p}} \cdot \left(\sum_{i=1}^n (x^i + y^i)^p\right)^{\frac{1}{q}} \end{aligned}$$

从而求和得到 $\sum_{i=1}^n (x^i + y^i)^p \leq \left(\left(\sum_{i=1}^n (x^i)^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^n (y^i)^p\right)^{\frac{1}{p}}\right) \cdot \left(\sum_{i=1}^n (x^i + y^i)^p\right)^{\frac{1}{q}}$ ，移项即得

$$\left(\sum_{i=1}^n (x^i)^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^n (y^i)^p\right)^{\frac{1}{p}} \geq \left(\sum_{i=1}^n (x^i + y^i)^p\right)^{1-\frac{1}{q}} = \left(\sum_{i=1}^n (x^i + y^i)^p\right)^{\frac{1}{p}}$$

$\|\vec{x}\|_p = \left(\sum_{i=1}^m |x^i|^p\right)^{\frac{1}{p}}$ 满足正定、正齐次、三角不等式，从而为范数 $\square$

5

设 $m > 1$ ，证明 $\mathbb{A} = \{\vec{x} \in \mathbb{R}^m \mid \|\vec{x}\| = 1\}$ 是道路连通集。

证明： $\forall \vec{x}, \vec{y} \in E$ ，令 $f(t) = \frac{(1-t)\vec{x} + t\vec{y}}{\|(1-t)\vec{x} + t\vec{y}\|}$ ，对 $\vec{x} = \vec{y}$ 时，显然若 $\vec{x} \neq \vec{y}$ ， $f(0) = \vec{x}$ ，

$f(1) = \vec{y}$ ，先证明该函数映射连续性。有线性函数 $g_1 : [0, 1] \rightarrow \mathbb{R}^m, t \mapsto (1-t)\vec{x} + t\vec{y}$ 连续，考虑函数 $g_2 : [0, 1] \rightarrow [0, 1], t \mapsto \|(1-t)\vec{x} + t\vec{y}\|$ ，由范数的正齐次性和三角不等式得

$$\forall t_1, t_2 \in [0, 1], |g_2(t_1) - g_2(t_2)| = \left| \|\vec{x} + t_1(\vec{y} - \vec{x})\| - \|\vec{x} + t_2(\vec{y} - \vec{x})\| \right| \leq |t_1 - t_2| \|\vec{y} - \vec{x}\|$$

而 $\|\vec{y} - \vec{x}\| \leq \|\vec{y}\| + \|\vec{x}\| = 2$ 为有限量，则有

$$\forall \epsilon > 0, \exists \delta < \frac{\epsilon}{2}, s.t. \forall t_1, t_2 \in [0, 1], |t_1 - t_2| < \delta, |g_2(t_1) - g_2(t_2)| \leq |t_1 - t_2| \|\vec{y} - \vec{x}\| < \epsilon$$

从而 g_2 为连续函数，同时由范数的正定性， $\|(1-t)\vec{x} + t\vec{y}\| \geq 0$ ，若 $\|(1-t)\vec{x} + t\vec{y}\| = 0$

由正定性 $(1-t)\vec{x} + t\vec{y} = 0$ 得 $\vec{x} = -\vec{y}$ ，只要 $\vec{x} \neq -\vec{y} \implies \|(1-t)\vec{x} + t\vec{y}\| \neq 0$

令 $g_3 : x \in (0, 1] \mapsto \frac{1}{x} \in [1, +\infty)$ 为连续函数，复合函数 $f(t) = (g_3 \circ g_2) \cdot g_1$ 也连续

而 $\|f(t)\| = \frac{\|(1-t)\vec{x} + t\vec{y}\|}{\|(1-t)\vec{x} + t\vec{y}\|} = 1$ ，则对 $t \in [0, 1]$ ，都有 $f(t) \in \mathbb{A}$ ，满足道路连通集条件

而针对特殊情况（两向量恰好反向）， $\vec{x} = -\vec{y}$ ，可选取 $\vec{z} \neq \vec{x}, \vec{z} \neq \vec{y}$ ，令 $\vec{x}$ 到 $\vec{z}$ 的连续映射为

$f_1(t)$ ， $\vec{z}$ 到 $\vec{y}$ 的连续映射为 $f_2(t)$ ，可以构造连续映射

$$f(t) = \begin{cases} f_1(2t), t \in [0, \frac{1}{2}) \\ f_2(2t-1), t \in [\frac{1}{2}, 1] \end{cases}$$

也满足道路连续性的条件，从而

$$\forall \vec{x}, \vec{y} \in \mathbb{A}, \exists \text{ 连续 } f(t), f(0) = \vec{x}, f(1) = \vec{y}, s.t. \forall t \in [0, 1], f(t) \in \mathbb{A}$$

即 $\mathbb{A}$ 为道路连通集 $\square$

6

设 $m > 1$ ，证明 $\mathbb{D} = \{A \mid A \text{ 是 } m \text{ 阶方阵}, \det A > 0\}$ 是道路连通集。

证明：首先证明单位阵 I 到 $\mathbb{D}$ 中任意元素 $A \in \mathbb{D}, \det A > 0$ 是连通的，对单位阵 I ，总存在一系列初等变换有

$$E_1 E_2 \cdots E_n I E'_1 E'_2 \cdots E'_n = A$$

若 E_i 是倍乘变换，倍乘系数为 $k \neq 0$ ，则对 $\forall B \in \mathbb{D}$ ，作连续映射

$$f_i(t) : \mathcal{GL}^+(m) \longrightarrow \mathcal{GL}^+(m) \quad B \longmapsto (1 + (k-1)t)E_i B$$

若 E_i 是倍加变换，倍加系数为 $m > 0$ ，则对 $\forall C \in \mathbb{D}$ ，作连续映射

$$g_i(t) : \mathcal{GL}^+(m) \longrightarrow \mathcal{GL}^+(m) \quad C \longmapsto (1 + (k-1)t)E_i C$$

若 E_i 是换行 / 换列变换并对其中一行 / 列乘以 -1 （保证行列式 > 0 ），对改变的两行 / 两列有连续复合映射

$$(A, B) \xrightarrow{E_1} (A+B, B) \xrightarrow{E_2} (A+B, -A) \xrightarrow{E_3} (B, -A) \xrightarrow{E_4(-1)} (B, A)$$

综上，对所有初等矩阵 E_i ，均存在连续映射 $f_{E_i}(t)$ 从 I 到 E_i ，使得 I 到 $A \in \mathbb{D}$ 是连通的

对 $\forall A_1, A_2 \in \mathbb{D}$ ，对 A_2 实行上述步骤得到连续映射 $I \longmapsto A_2, F(t) : \prod_{i=1}^n f_{E_i}$ ，则 $A_1^{-1} \circ F$ 满足 $A_1 \longmapsto A_2$ 连续映射，且由于 $f_i(t)$ 和 $g_i(t)$ 都映射到 $\mathbb{D}$ 中， $\forall t \in [0, 1], A_1^{-1} \circ F(t) \in \mathbb{D}$

，即 $\mathbb{D}$ 道路连通 $\square$

7

设 $f : \mathbb{R}^m \setminus \{0\} \longmapsto \mathbb{R}$ ，且 f 连续，满足 $\forall t > 0, \vec{x} \neq 0, f(t\vec{x}) = f(\vec{x})$ ，证明 f 有最大值和最小值。

证明：令单位“实心球”集合 $\mathbb{S} = \{\vec{y} \mid \|\vec{y}\| \leq 1\}$ 为有界闭集，对 $\forall \vec{y} \in \mathbb{R}^m \setminus \{0\}$ ，若 $\|\vec{y}\| \leq 1$ ，

则 $\vec{y} \in \mathbb{S}$ ，否则 $\|\vec{y}\| > 1$ ，令 $\vec{z} = \frac{\vec{y}}{\|\vec{y}\|}$ ，其范数 $\|\vec{z}\| = 1, \vec{z} \in \mathbb{S}$ ， $f(\vec{y}) = f(\|\vec{y}\| \cdot \vec{z}) = f(\vec{z})$

定义映射 $g: \mathbb{S} \rightarrow \mathbb{R}, g(\vec{x}) = f(\vec{x})$, 由知 f 和 g 的值域相同, 且 g 为有界闭集上的连续映射
从而 f 和 g 都有最大值和最小值 □

8

设 $f(x, y, z) = \frac{x + 2y + 3z}{x^2 + y^2 + z^2 + 1}$, 证明 f 有正的最大值和负的最小值。

证明: 令三维欧式空间向量 $\vec{r} = (x, y, z)$, 有

$$f(x, y, z) = f(\vec{r}) = \frac{\vec{r} \cdot (1, 2, 3)}{\|\vec{r}\|^2 + 1} = \frac{\|\vec{r}\| \cdot \sqrt{14} \cos \theta}{\|\vec{r}\|^2 + 1}$$

由均值不等式 $\frac{\|\vec{r}\|}{\|\vec{r}\|^2 + 1} \leq \frac{1}{2}$, 以及 $\cos \theta \in [-1, 1]$, 得 $f(\vec{r}) \in [-\frac{\sqrt{14}}{2}, \frac{\sqrt{14}}{2}]$ 有界

而 $f(x, y, z)$ 为连续函数, 从而有最值, 且当 $x, y, z > 0$ 时 $f(x, y, z) > 0$; 当 $x, y, z < 0$ 时

$f(x, y, z) < 0$, 则 f 有正的最大值和负的最小值 □

微积分第二次作业 ($W \times f$) 2022.3.4

1

1、(连续性与偏连续性) 设 $f: [a, b] \times [c, d] \rightarrow \mathbb{R}$ 。

(a) 如果 f 是连续函数, 那么是否成立: 对任意 $y \in [c, d], f(\cdot, y): [a, b] \rightarrow \mathbb{R}$ 连续; 对任意 $x \in [a, b]$

$f(x, \cdot): [c, d] \rightarrow \mathbb{R}$ 连续?

(b) 如果对任意 $y \in [c, d], f(\cdot, y): [a, b] \rightarrow \mathbb{R}$ 连续; 对任意 $x \in [a, b], f(x, \cdot): [c, d] \rightarrow \mathbb{R}$ 连续, 那么 f 是否为连续函数? 考虑

$$f(x, y) = \begin{cases} \frac{xy}{x^2 + y^2}, & xy \neq 0 \\ 0, & xy = 0 \end{cases}$$

(c) 设 $x_0 \in [a, b], y_0 \in [c, d], f$ 满足: $f(\cdot, y_0): [a, b] \rightarrow \mathbb{R}$ 连续; 并且对任意 $\varepsilon > 0$, 存在

$\delta(\varepsilon) > 0$ 使得对任意 $x \in [a, b]$ 以及任意 $y \in [c, d]$, 只要 $|y - y_0| < \delta(\varepsilon)$, 就有

$|f(x, y) - f(x, y_0)| < \varepsilon$ 。证明 f 在 (x_0, y_0) 连续。

(d) 设 $f(x, y)$ 分别对每个变量 x 和 y 是连续的, 且对 y 是单调不减的, 证明 f 关于 (x, y) 是连续的。

解: (a) 是。由连续函数定义, 对 $\mathbf{x}^* = (x^1, x^2)$

$$\forall \mathbf{x}^* \in [a, b] \times [c, d], \forall \epsilon > 0, \exists \delta > 0, s. t. \forall \|x - x^*\| \leq \delta, \|f(\mathbf{x}) - f(\mathbf{x}^*)\| < \epsilon$$

则固定 x^1 时, 仅取 $\|x - x^*\| \leq \delta$ 中 x 第一分量为 x^1 的点集, 由上述定义知

$$\forall \text{ 固定 } x^1 \in [a, b], \forall x^2 \in [c, d], \mathbf{x}^* = (x^1, x^2), s. t. \forall \epsilon > 0, \exists \delta > 0, \\ \forall \|x - x^*\| \leq \delta, \|f(\mathbf{x}) - f(\mathbf{x}^*)\| < \epsilon$$

从而固定 x^1 时, $f(x^1, x^2)$ 随 x^2 连续; 由对称性可得, 固定 x^2 时, $f(x^1, x^2)$ 随 x^1 连续

$$(b) \text{ 否。考虑函数 } f(x, y) = \begin{cases} \frac{xy}{x^2 + y^2}, & xy \neq 0 \\ 0 & xy = 0 \end{cases} \quad x, y \in [-1, 1], \text{ 由对称性, 固定 } x \text{ 有}$$

若 $x = 0$, 则 $f(x, y) = 0$ 在 $y \in [-1, 1]$ 区间上均满足 $f(0, y)$ 随 y 连续;

若 $x \neq 0$, 则函数 $g_1(y) = x^2 + y^2 \neq 0$, $g_2(z) = \frac{1}{z} (z \neq 0)$, $g_3(y) = xy$ 三者均连续

则 $f(x, y) = (g_2 \circ g_1) \cdot g_3$ 复合函数连续, 即 $\forall x \in [-1, 1], f(x, \cdot) : [c, d] \rightarrow \mathbb{R}$ 连续

同理 $\forall y \in [-1, 1], f(\cdot, y) : [c, d] \rightarrow \mathbb{R}$ 连续, 而考虑 $f(x, y)$ 在原点处的连续性

$$\lim_{(x, y) \rightarrow (0, 0)} \frac{xy}{x^2 + y^2} \Big|_{(x, y) \rightarrow (x, kx)} = \frac{k}{1 + k^2} \in [-\frac{1}{2}, \frac{1}{2}]$$

k 值变化时, $f(x, y) = \frac{xy}{x^2 + y^2}$ 在 $(0, 0)$ 处的极限不存在, 从而 f 不一定为连续函数

(c) 证明: 由 f 满足 $f(\cdot, y_0) : [a, b] \rightarrow \mathbb{R}$ 连续, 翻译成 $\epsilon - \delta$ 语言有

$$\forall y_0 \in [c, d], \forall \epsilon > 0, \exists \delta_1 > 0, |x - x_0| < \delta_1(\epsilon), |f(x_0, y_0) - f(x, y_0)| < \epsilon$$

由 ϵ 的任意性, 则 $\forall \epsilon' > 0$, 上述取 $\epsilon = \frac{1}{2}\epsilon'$, 另一条件为

$$\forall \epsilon > 0, \exists \delta_2 > 0, \forall x \in [a, b], y \in [c, d], |y - y_0| < \delta_2(\epsilon), |f(x, y) - f(x, y_0)| < \epsilon$$

由后者条件中 x 的任意性, 取 $\epsilon = \frac{1}{2}\epsilon'$, 则考虑欧几里得空间中的 2-范数有

$$\forall \epsilon' > 0, \exists \delta = \min(\delta_1, \delta_2) > 0, \forall x \in [a, b], y \in [c, d]$$

$$\|x - x^*\| = \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta \implies |x - x_0| < \delta_1, |y - y_0| < \delta_2$$

$$|f(x, y) - f(x_0, y_0)| \leq |f(x, y) - f(x, y_0)| + |f(x_0, y_0) - f(x, y_0)| < \frac{1}{2}\epsilon' + \frac{1}{2}\epsilon' = \epsilon'$$

则 $f(x, y)$ 在 (x_0, y_0) 处连续

(d) 证明: 由于 $f(x, y)$ 对 y 单调不减, 则有对 $\forall x \in [a, b]$, 不妨只考虑 $y > y_0$ 的部分

$\delta(y_0) > 0, 0 < f(x, y_0 + \delta(y_0)) - f(x, y_0) < \epsilon$, 取 $y = y_0 + \delta(y_0)$, 考虑

$$\begin{aligned} |f(x, y) - f(x_0, y_0)| &= |f(x, y_0 + \delta(y_0)) - f(x_0, y_0)| \\ &\leq |f(x, y_0 + \delta(y_0)) - f(x, y_0)| + |f(x, y_0) - f(x_0, y_0)| < \epsilon + \epsilon = 2\epsilon \end{aligned}$$

其中后者由于 $f(x, y)$ 对 x 连续可以保证, 从而 $f(x, y)$ 在 (x_0, y_0) 处连续

$$A = \lim_{(x,y) \rightarrow (a,b)} f(x,y), \quad B = \lim_{y \rightarrow b} \lim_{x \rightarrow a} f(x,y)$$

都存在。证明 $A = B$ 。由此知道, 如果

$$\lim_{y \rightarrow b} \lim_{x \rightarrow a} f(x,y) \neq \lim_{x \rightarrow a} \lim_{y \rightarrow b} f(x,y)$$

则极限 $\lim_{(x,y) \rightarrow (a,b)} f(x,y)$ 不存在。

证明: 由于对 $\forall 0 < |y - b| < \delta_0$, $\varphi(y) = \lim_{x \rightarrow a} f(x,y)$ 存在, 即

$$\forall \epsilon > 0, y \in U, \exists \delta_1(y), s. t. |x - a| < \delta_1(y), |f(x,y) - \varphi(y)| < \epsilon$$

而又有 $\lim_{y \rightarrow b} \varphi(y) = B$ 存在, 即 $\forall \epsilon > 0, \exists \delta_2 > 0, |y - b| < \delta_2, |\varphi(y) - B| < \epsilon$

而 $\lim_{(x,y) \rightarrow (a,b)} f(x,y) = A$, 即有

$$\forall \epsilon > 0, \exists \delta_0 > 0, \forall x, y \in U, \|v^* - v\| < \delta_0(\epsilon), |f(x,y) - A| < \epsilon$$

则取 $|y - b| < \min(\delta_0, \delta_2), |x - a| < \min(\delta_1(y), \delta_0)$, 总有 $|f(x,y) - B| < \epsilon + \epsilon = 2\epsilon$

从而 $|A - B| < \epsilon + 2\epsilon = 3\epsilon$, 则 $A = B$, 且若 $\lim_{y \rightarrow b} \lim_{x \rightarrow a} f(x,y) \neq \lim_{x \rightarrow a} \lim_{y \rightarrow b} f(x,y)$ 两者极限存在

反证法, 若极限 $\lim_{(x,y) \rightarrow (a,b)} f(x,y)$ 存在, 则由上述证明以及对称性知, 三者相等, 与假设矛盾, 从而

极限 $\lim_{(x,y) \rightarrow (a,b)} f(x,y)$ 不存在

3

设

(a) 极限 $H(y) = \lim_{x \rightarrow a} f(x,y)$ 对任意 $y \neq b$ 存在;

(b) 极限 $G(x) = \lim_{y \rightarrow b} f(x,y)$ 对任意 x 存在, 且存在 $\delta_0 > 0$ 使得: 对任意 $\epsilon > 0$, 存在

$\delta_2(\epsilon) > 0$ 对任意 $0 < |y - b| < \delta_2(\epsilon)$ 任意 $0 < |x - a| < \delta_0$ 有 $|f(x,y) - G(x)| < \epsilon$, 则

极限 $\lim_{x \rightarrow a} \lim_{y \rightarrow b} f(x,y), \lim_{y \rightarrow b} \lim_{x \rightarrow a} f(x,y)$ 和 $\lim_{x \rightarrow a, y \rightarrow b} f(x,y)$ 都存在, 且

$$\lim_{x \rightarrow a, y \rightarrow b} f(x,y) = \lim_{y \rightarrow b} \lim_{x \rightarrow a} f(x,y) = \lim_{x \rightarrow a} \lim_{y \rightarrow b} f(x,y)$$

证明: (Cauchy准则) $\forall 0 < |y_1 - b| < \delta_2(\epsilon), 0 < |y_2 - b| < \delta_2(\epsilon)$, 对 $|x - a| < \delta_0$, 都有

$$|f(x, y_1) - G(x)| < \frac{\epsilon}{2}, |f(x, y_2) - G(x)| < \frac{\epsilon}{2} \therefore |f(x, y_1) - f(x, y_2)| < \frac{\epsilon}{2} \cdot 2 = \epsilon$$

则令 $x \rightarrow a$, 由于 $H(y) = \lim_{x \rightarrow a} f(x,y)$ 存在, 则有 $|H(y_1) - H(y_2)| < \epsilon$, 即

$$\forall \epsilon > 0, \exists \delta > 0, \forall y_i (i = 1, 2), 0 < |y_i - b| < \delta, |H(y_1) - H(y_2)| < \epsilon$$

由柯西收敛准则得到 $\lim_{y \rightarrow b} \lim_{x \rightarrow a} f(x,y)$ 极限存在, 由 $0 < |y_0 - b| < \delta_2(\epsilon)$

$$|G(x_1) - G(x_2)| \leq |G(x_1) - f(x_1, y_0)| + |G(x_2) - f(x_2, y_0)| + |f(x_1, y_0) - f(x_2, y_0)|$$

只需要 $0 < |x - a| < \delta_0(\frac{1}{3}\epsilon)$, 就有 $|G(x_1) - f(x_1, y_0)| < \frac{1}{3}\epsilon, |G(x_2) - f(x_2, y_0)| < \frac{1}{3}\epsilon$

而由于 $H(y) = \lim_{x \rightarrow a} f(x, y)$ 对 $y \neq b$ 时均存在, 则添加条件 $0 < |x - a| < \delta_1(\frac{1}{3}\epsilon)$

就有 $|f(x_1, y_0) - f(x_2, y_0)| < \frac{1}{3}\epsilon$, 则满足下列条件

$$0 < |x - a| < \min(\delta_0(\frac{1}{3}\epsilon), \delta_1(\frac{1}{3}\epsilon)), 0 < |y - b| < \delta_1(\epsilon), |G(x_1) - G(x_2)| < \frac{1}{3}\epsilon \cdot 3 = \epsilon$$

由柯西收敛准则得 $\lim_{x \rightarrow a} \lim_{y \rightarrow b} f(x, y)$ 存在, 由作业第2题知, 三种极限存在且相等

4

设

$$f(x, y) = \begin{cases} x \sin\left(\frac{1}{y}\right), & y \neq 0 \\ 0, & y = 0 \end{cases}$$

讨论极限 $\lim_{(x,y) \rightarrow (0,0)} f(x, y), \lim_{y \rightarrow 0} \lim_{x \rightarrow 0} f(x, y), \lim_{x \rightarrow 0} \lim_{y \rightarrow 0} f(x, y)$

解: 显然 $\lim_{x \rightarrow 0} f(x, y) = 0$, 则有 $\lim_{y \rightarrow 0} \lim_{x \rightarrow 0} f(x, y) = 0$, 而对 $y \rightarrow 0$ 的性质有

$$\forall |x| > 0, f(x, y) = \begin{cases} x \sin\left(\frac{1}{y}\right), & y \neq 0 \\ 0, & y = 0 \end{cases}$$

对 $y_n = \frac{1}{2n\pi + \frac{\pi}{2}}, y_m = \frac{1}{2m\pi - \frac{\pi}{2}}$, 总有 $f(x, y_n) - f(x, y_m) = x - (-x) = 2x$

由柯西收敛准则, $\forall x \neq 0, \lim_{y \rightarrow 0} x \sin\left(\frac{1}{y}\right)$ 不存在, 从而 $\lim_{x \rightarrow 0} \lim_{y \rightarrow 0} f(x, y)$ 极限不存在

而对 $\forall \epsilon > 0, \exists \delta = \epsilon, \sqrt{x^2 + y^2} < \delta, |f(x, y) - 0| \leq |x| \left| \sin\left(\frac{1}{y}\right) \right| \leq |x| < \delta = \epsilon$

从而 $\lim_{(x,y) \rightarrow (0,0)} f(x, y)$ 极限存在, 且极限为 0

5

判断以下极限的存在性; 若存在, 求出极限值。

$$\begin{aligned} (1) \quad \lim_{(x,y) \rightarrow (0,0)} \frac{\arcsin(x^2 + y^2)}{x^2 + y^2} \quad (3) \quad \lim_{(x,y) \rightarrow (0,0)} (x^2 + y^2)e^{x-y} \quad (5) \quad \lim_{(x,y) \rightarrow (0,0)} \frac{x+y}{|x| + |y|} \\ (7) \quad \lim_{(x,y) \rightarrow \infty} \frac{x+y}{x^2 + xy + y^2} \quad (9) \quad \lim_{(x,y) \rightarrow \infty} \left(\frac{|xy|}{x^2 + xy + y^2} \right)^{x^2} \end{aligned}$$

解: (1) 令 $z = x^2 + y^2$, $\lim_{z \rightarrow 0} \frac{\arcsin(z)}{z} = 1$, 而当 $x, y \rightarrow 0$ 时, $z = x^2 + y^2 \neq 0$ 则

$$\lim_{(x,y) \rightarrow (0,0)} \frac{\arcsin(x^2 + y^2)}{x^2 + y^2} = 0$$

(3) 由极限的连续性, 且 $\lim_{(x,y) \rightarrow (0,0)} x^2 + y^2 = 0$, $\lim_{(x,y) \rightarrow (0,0)} e^{x-y} = 0$ 则极限存在, 且极限为 0

(5) 只需取 $k > 0$, $\lim_{(x,kx) \rightarrow (0,0)} \frac{x+y}{|x|+|y|} = \frac{(1+k)x}{(1+k)|x|}$ 从而极限不存在

(7) 放缩有 $|\frac{x+y}{x^2+xy+y^2}| \leq \frac{|x+y|}{|x^2+y^2|-|xy|} \leq \frac{|x+y|}{|2xy|-|xy|} \leq \frac{|x|+|y|}{|xy|} \leq \frac{1}{|x|} + \frac{1}{|y|}$

从而 $\lim_{(x,y) \rightarrow \infty} \frac{x+y}{x^2+xy+y^2} = 0$

(9) 若取 $y = x$, 有 $\lim_{(x,x) \rightarrow \infty} \frac{|xy|}{x^2+xy+y^2} = \pm \frac{1}{3}$, 此时 $\lim_{x \rightarrow \infty} (\pm \frac{1}{3})^{x^2} = 0$

又若取 $y = -x$, 有 $\lim_{(x,x) \rightarrow \infty} \frac{|xy|}{x^2+xy+y^2} = \pm 1$, 此时 $\lim_{x \rightarrow \infty} (\pm \frac{1}{3})^{x^2} \neq 0$

从而 $\lim_{(x,y) \rightarrow \infty} \left(\frac{|xy|}{x^2+xy+y^2} \right)^{x^2}$ 极限不存在

6

设 $\Omega \subseteq \mathbb{R}^m$ 是一个非空集合。

(a) 证明函数 $f: \mathbb{R}^m \rightarrow \mathbb{R}, f(\mathbf{x}) = \inf_{\mathbf{y} \in \Omega} \|\mathbf{y} - \mathbf{x}\|$, 是一个连续函数。

(b) 若 Ω 是闭集, 证明对任意 $\mathbf{x}$ 不在 Ω 中, 存在 $\mathbf{y}^* \in \Omega$ 使得 $f(\mathbf{x}) = \|\mathbf{y}^* - \mathbf{x}\|$, 并且 $\mathbf{y}^* \in \partial\Omega$ (即 $\mathbf{y}^*$ 的任意邻域中都有不属于 Ω 的点)。

(c) 若 Ω_1, Ω_2 是闭集且其中至少一个有界, 证明存在 $\mathbf{x}_k^* \in \Omega_k$ 使得

$$\|\mathbf{x}_1^* - \mathbf{x}_2^*\| = \inf_{\mathbf{x}_1 \in \Omega_1, \mathbf{x}_2 \in \Omega_2} \|\mathbf{x}_1 - \mathbf{x}_2\|.$$

(d) 举例说明上述“闭”、“有界”的条件是不能或缺的。

证明: (a) $\forall \epsilon > 0, \exists \delta > 0, \|\mathbf{x} - \mathbf{x}_0\| < \delta$, 范数三角不等式 $\|\mathbf{y} - \mathbf{x}\| \leq \|\mathbf{y} - \mathbf{x}_0\| + \|\mathbf{x}_0 - \mathbf{x}\|$

左侧放缩 $\inf_{\mathbf{y} \in \Omega} \|\mathbf{y} - \mathbf{x}\| \leq \|\mathbf{y} - \mathbf{x}\| \leq \|\mathbf{y} - \mathbf{x}_0\| + \|\mathbf{x}_0 - \mathbf{x}\|$ 对所有的 $\mathbf{y}$ 成立, 则取特定 $\mathbf{y}$ 得

$$\inf_{\mathbf{y} \in \Omega} \|\mathbf{y} - \mathbf{x}\| \leq \inf_{\mathbf{y} \in \Omega} \|\mathbf{y} - \mathbf{x}_0\| + \|\mathbf{x}_0 - \mathbf{x}\|$$

同理交换 $\mathbf{x}$ 和 $\mathbf{x}_0$ 得到 $|\inf_{\mathbf{y} \in \Omega} \|\mathbf{y} - \mathbf{x}\| - \inf_{\mathbf{y} \in \Omega} \|\mathbf{y} - \mathbf{x}_0\|| \leq \|\mathbf{x} - \mathbf{x}_0\|$, 则取 $\delta = \epsilon$, 就有

$$\forall \epsilon > 0, \exists \delta = \epsilon, \|\mathbf{x} - \mathbf{x}_0\| < \epsilon, |f(\mathbf{x}) - f(\mathbf{x}_0)| \leq \delta = \epsilon$$

则 $f(\mathbf{x}) = \inf_{\mathbf{y} \in \Omega} \|\mathbf{y} - \mathbf{x}\|$ 连续

(b) 由三角不等式 $\|\mathbf{y}^* - \mathbf{y}_n\| + \|\mathbf{y}_n - \mathbf{x}\| \geq \|\mathbf{y}^* - \mathbf{x}\|$ 若 $f(\mathbf{x})$ 中 $\mathbf{y}$ 的取值为 Ω 的内点, 则

$\forall \epsilon_0 > \epsilon > 0, \exists \|\mathbf{y}^* - \mathbf{y}_n\| < \epsilon$, 从而可以找到其领域中的 $\mathbf{y}_n$ 并尽量使三者共线, 从而矛盾

从而 $\mathbf{y}^* \in \partial\Omega$

(c) 假设 Ω_2 为有界集合, 固定在 Ω_1 中一点, 由在有界闭集上的连续函数有界有

$$\forall \mathbf{x}_1 \in \Omega_1, \exists \mathbf{x}_2^* \in \Omega_2, \|\mathbf{x}_1 - \mathbf{x}_2^*\| = \inf_{\mathbf{x}_2 \in \Omega_2} \|\mathbf{x}_2 - \mathbf{x}_1\|$$

则取所有 $\mathbf{x}_1$ 中范数最小的, 则存在 $\|\mathbf{x}_1^* - \mathbf{x}_2^*\| = \inf_{\mathbf{x}_1 \in \Omega_1, \mathbf{x}_2 \in \Omega_2} \|\mathbf{x}_1 - \mathbf{x}_2\|$.

(d) 取 $\Omega: x^2 + y^2 < 1$, 范数定义为二维平面中的欧式距离, 则 Ω 与 $(2, 0)$ 就没有最小值

而两个无界区域之间有可能没有范数最小值, 如 $\{(x, y) | y = \tan(x), 0 \leq x < \frac{\pi}{2}\}$

和 $\{(x, y) | x = \frac{\pi}{2}\}$, 两者的欧式距离没有最小值 (无限趋于 0)

微积分第三次作业 ($W \times f$) 2022. π

教材习题1.4.4

求下列函数的全微分

$$(1) u = \sin \frac{1}{\sqrt{x^2 + y^2 + z^2}} \text{ at } \left(\frac{\sqrt{2}}{2}, \frac{1}{2}, -\frac{1}{2}\right) \quad (2) \arccos e^{xy}$$

解: (1) $du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz$, 代入有

$$\begin{aligned} du &= \cos \frac{1}{\sqrt{x^2 + y^2 + z^2}} \cdot \left(-\frac{1}{2}(x^2 + y^2 + z^2)^{-\frac{3}{2}}\right)(2x dx + 2y dy + 2z dz) \\ &= -\cos 1 \cdot (x dx + y dy + z dz) = -\cos 1 \cdot \left(\frac{\sqrt{2}}{2} dx + \frac{1}{2} dy - \frac{1}{2} dz\right) \end{aligned}$$

$$(2) \text{ 由 } (\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$$

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy = -\frac{1}{\sqrt{1-e^{2xy}}} (ye^{xy} dx + xe^{xy} dy) = -\frac{1}{\sqrt{e^{-2xy}-1}} (y dx + x dy)$$

教材习题1.4.6

已知扇形中心角 $\alpha = 60^\circ$, 半径 $R = 20$ cm, 当 α 增加 1° 时, 为使扇形面积保持不变, 其半径的增加量 ΔR 近似等于多少?

解：由面积公式 $S = \frac{1}{2}R^2\alpha$ ，要求 $dS = \frac{1}{2}R^2d\alpha + R\alpha dR = 0$ 取 $dR = \Delta R, d\alpha = \Delta\alpha$

解得 $\Delta R = -\frac{R}{2} \frac{\Delta\alpha}{\alpha} = -\frac{1}{6} \text{ cm}$

$\left| \frac{1}{6} \approx 0.16666667 \right.$ ，与直接计算 $|\Delta R| = 0.16461185 \text{ cm}$ 相对误差为 1.2%

教材习题1.4.11(4)

求下列函数在点 P_0 处沿方向 l 的方向导数。

(4) $z = \ln(x_1 + x_2 + \cdots + x_n), P_n = (1, 0, \cdots, 0), l = (1, 1, \cdots, 1)$

解：由于方向导数针对单位向量，先单位化 $l_{\text{unit}} = \frac{1}{\sqrt{n}}(1, 1, \cdots, 1)$ 则由方向导数定义

$$\partial_{\vec{l}} z = \frac{d}{dt} \ln \left(1 + n \cdot \frac{t}{\sqrt{n}} \right) \Big|_{t=0} = \sqrt{n}$$

教材习题1.4.12(2)

求下列数量场的梯度。(2) $u(x, y, z) = \frac{xyz}{x + y + z}$

解：代入标准正交基 x, y, z 下的梯度

$$\nabla u = \frac{\partial u}{\partial x} \hat{x} + \frac{\partial u}{\partial y} \hat{y} + \frac{\partial u}{\partial z} \hat{z} = \frac{yz(y+z)}{(x+y+z)^2} \hat{x} + \frac{xz(x+z)}{(x+y+z)^2} \hat{y} + \frac{xy(x+y)}{(x+y+z)^2} \hat{z}$$

教材习题1.4.15

证明下列函数满足相应的等式。

(3) $\begin{cases} u = e^x \cos y \\ v = e^x \sin y \end{cases}$ 满足 Cauchy-Riemann 条件 $\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \end{cases}$ ，且分别满足 $\Delta u = 0$

(4) $n > 2$, $u = (\sqrt{x_1^2 + x_2^2 + \cdots + x_n^2})^{2-n}$ 满足 $\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2} = 0$

证明：(3) 代入有 $\frac{\partial u}{\partial x} = e^x \cos y = \frac{\partial v}{\partial y} = e^x \cos y, \frac{\partial u}{\partial y} = -e^x \sin y = -\frac{\partial v}{\partial x}$

则若 $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$, $f(u, v) = f(e^x \cos y, e^x \sin y)$ 代入计算两次偏导数有

$$\begin{aligned}\frac{\partial f}{\partial x} &= \frac{\partial f}{\partial u} e^x \cos y + \frac{\partial f}{\partial v} e^x \sin y = u \frac{\partial f}{\partial u} + v \frac{\partial f}{\partial v} \\ \frac{\partial f}{\partial y} &= -\frac{\partial f}{\partial u} e^x \sin y + \frac{\partial f}{\partial v} e^x \cos y = -v \frac{\partial f}{\partial u} + u \frac{\partial f}{\partial v}\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 f}{\partial x^2} &= u \frac{\partial f}{\partial u} + u^2 \frac{\partial^2 f}{\partial u^2} + v \frac{\partial f}{\partial v} + v^2 \frac{\partial^2 f}{\partial v^2} \\ \frac{\partial^2 f}{\partial y^2} &= -u \frac{\partial f}{\partial u} + v^2 \frac{\partial^2 f}{\partial u^2} - v \frac{\partial f}{\partial v} + u^2 \frac{\partial^2 f}{\partial v^2}\end{aligned}$$

$$\text{从而 } \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0 \implies (u^2 + v^2) \left(\frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2} \right) = 0 \implies \frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2} = 0$$

$$(4) \text{ 对 } \forall i \in [1, n], \text{ 有 } \frac{\partial u}{\partial x_i} = \frac{(1 - \frac{n}{2}) \cdot 2x_i}{(x_1^2 + \cdots + x_n^2)^{\frac{n}{2}}} = \frac{(2 - n) \cdot x_i}{(x_1^2 + \cdots + x_n^2)^{\frac{n}{2}}}$$

$$\text{则二阶导 } \frac{\partial^2 u}{\partial x_i^2} = \frac{2 - n}{(x_1^2 + \cdots + x_n^2)^{\frac{n}{2}}} + \frac{(n - 2) \cdot \frac{n}{2} x_i \cdot 2x_i}{(x_1^2 + \cdots + x_n^2)^{\frac{n}{2} + 1}}, \text{ 则对所有二阶导求和}$$

$$\sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2} = (x_1^2 + \cdots + x_n^2)^{-\frac{n}{2}} ((2 - n) \cdot n + (n - 2) \cdot n) = 0$$

教材习题1.5.3(3)

求下列复合函数的偏导数 $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ (已知 f 为可微函数)

$$(3) z = f(x^2 - y^2, e^{xy})$$

解: 令 $u = x^2 - y^2, v = e^{xy}$ 则代入计算偏导数

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot 2x + \frac{\partial z}{\partial v} \cdot ye^{xy}, \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot (-2y) + \frac{\partial z}{\partial v} \cdot xe^{xy}$$

教材习题1.5.5

已知函数 $u = f(x, y)$, 其中 $x = r \cos \theta, y = r \sin \theta$, f 可微, 证明

$$\left(\frac{\partial u}{\partial r} \right)^2 + \left(\frac{1}{r} \frac{\partial u}{\partial \theta} \right)^2 = \left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2$$

证明: 对 $u = f(x, y) = f(r \cos \theta, r \sin \theta)$ 对 r, θ 分别求偏导有

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} \cdot \cos \theta + \frac{\partial u}{\partial y} \cdot \sin \theta, \frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x} \cdot (-r \sin \theta) + \frac{\partial u}{\partial y} \cdot (r \cos \theta)$$

$$\text{则计算 } \left(\frac{\partial u}{\partial r} \right)^2 + \left(\frac{1}{r} \frac{\partial u}{\partial \theta} \right)^2 = \left(\frac{\partial u}{\partial x} \cdot \cos \theta + \frac{\partial u}{\partial y} \cdot \sin \theta \right)^2 + \left(\frac{\partial u}{\partial x} \cdot (-\sin \theta) + \frac{\partial u}{\partial y} \cdot (\cos \theta) \right)^2$$

$$= \left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2 \text{ 从而 } \left(\frac{\partial u}{\partial r} \right)^2 + \left(\frac{1}{r} \frac{\partial u}{\partial \theta} \right)^2 = \left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2$$

教材习题1.5.7

设 $f \in C^2(\mathbb{R}^2)$ 满足 Laplace 方程 $(\frac{\partial f}{\partial x})^2 + (\frac{\partial f}{\partial y})^2 = 0$, 证明:

$$u(x, y) = f(\frac{x}{x^2 + y^2}, \frac{y}{x^2 + y^2})$$

也满足 Laplace 方程。

证明: (使用平面极坐标下拉普拉斯算子简化计算) $\Delta f_{\text{polar}} = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r} \frac{\partial f}{\partial r} + \frac{1}{r^2} \frac{\partial^2 f}{\partial \theta^2}$

则对于满足直角坐标系下 Laplace 方程 $\Delta f_{\text{rectan}} = 0 = \Delta f_{\text{polar}}$, 转换 $u(x, y) = f(\frac{1}{r}, \theta)$

$$\text{而 } \frac{\partial u}{\partial(\frac{1}{r})} = -r^2 \frac{\partial u}{\partial r}, \quad \frac{\partial^2 u}{\partial(\frac{1}{r})^2} = (-r^2)(-2r \frac{\partial u}{\partial r} - r^2 \frac{\partial^2 u}{\partial r^2}) = r^3(2 \frac{\partial u}{\partial r} + r \frac{\partial^2 u}{\partial r^2})$$

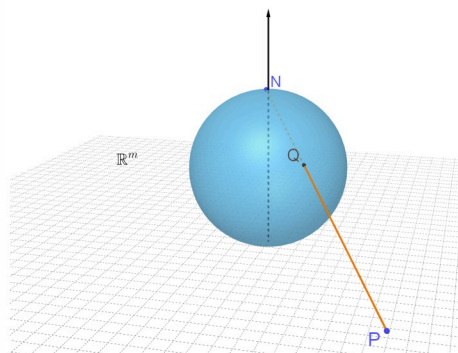
则代入以 $(\frac{1}{r}, \theta)$ 为度量的极坐标拉普拉斯算子判定有

$$\begin{aligned} \Delta f_{\text{polar}}' &= \frac{\partial^2 f}{\partial(\frac{1}{r})^2} + \frac{1}{\frac{1}{r}} \frac{\partial f}{\partial(\frac{1}{r})} + \frac{1}{(\frac{1}{r})^2} \frac{\partial^2 f}{\partial \theta^2} \\ &= r^3(2 \frac{\partial f}{\partial r} + r \frac{\partial^2 f}{\partial r^2}) + r(-r^2 \frac{\partial f}{\partial r}) + r^2 \frac{\partial^2 f}{\partial \theta^2} = r^3(\frac{\partial f}{\partial r} + r \frac{\partial^2 f}{\partial r^2}) + r^2 \frac{\partial^2 f}{\partial \theta^2} = r^4 \Delta f_{\text{polar}} \end{aligned}$$

从而 $\Delta f_{\text{polar}}' = 0$ 即 $f(\frac{1}{r}, \theta) = f(\frac{x}{x^2 + y^2}, \frac{y}{x^2 + y^2})$ 满足 Laplace 方程

讲义习题2.2.8

记 $\mathbb{R}^m \times \mathbb{R} = \{(X^1, \dots, X^m, X^{m+1}) \mid X^k \in \mathbb{R}\}$, $\mathbb{R}^m = \{(x^1, \dots, x^m) \mid x^k \in \mathbb{R}\}$



$N = (0, \dots, 0, 2) \in \mathbb{R}^m \times \mathbb{R}$, $S^m = \{(X^1, \dots, X^m, X^{m+1}) \mid (X^1)^2 + \dots + (X^m)^2 + (X^{m+1} - 1)^2 = 1\}$ 。定义球极投影, 对 $(x^1, \dots, x^m) \in \mathbb{R}^m$, 连接 $(x^1, \dots, x^m, 0)$ 与 N 的直线交 S^m 于 $(X^1, \dots, X^m, X^{m+1})$ 。证明映射

$$(X^1(x^1, \dots, x^m), \dots, X^m(x^1, \dots, x^m), X^{m+1}(x^1, \dots, x^m))$$

是可微映射, 并求它的 Jacobi 矩阵。

解: 连接 $N = (0, \dots, 0, 2)$ 与 $\mathbf{x} = (x^1, \dots, x^m, 0)$ 的直线参数方程为

$$\vec{l} = (0, \dots, 0, 2) + t(x^1, \dots, x^m, -2) = (tx^1, tx^2, \dots, tx^m, 2 - 2t)$$

$$\text{代入“高维球”方程得 } t^2 \sum_{i=1}^n (x^i)^2 + (1 - 2t)^2 = 1 \implies t = \frac{4}{4 + \sum_{i=0}^n (x^i)^2}$$

$$\text{则球极投影 } f(\mathbf{x}) = \left(\frac{4x^1}{4 + \sum_{i=0}^n (x^i)^2}, \frac{4x^2}{4 + \sum_{i=0}^n (x^i)^2}, \dots, \frac{4x^m}{4 + \sum_{i=0}^n (x^i)^2}, \frac{2 \sum_{i=0}^n (x^i)^2}{4 + \sum_{i=0}^n (x^i)^2} \right)$$

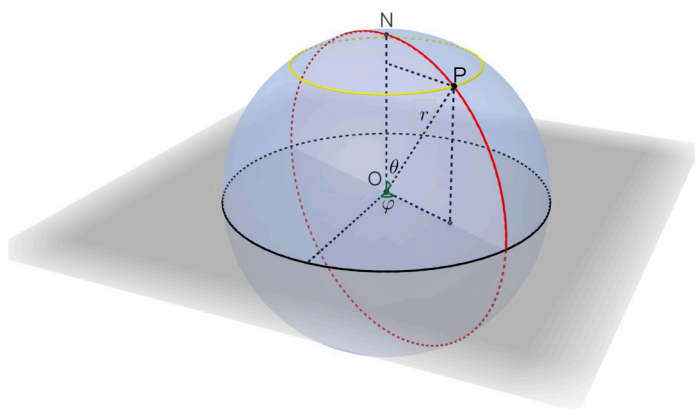
这是由于每个分量对 x^i 均是初等函数，从而该球极投影映射为可微映射

$$\text{雅可比矩阵定义 } \begin{bmatrix} \frac{\partial X_1}{\partial x_1} & \cdots & \frac{\partial X_1}{\partial x_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial X_{m+1}}{\partial x_1} & \cdots & \frac{\partial X_{m+1}}{\partial x_m} \end{bmatrix}, \text{ 记该矩阵第 } k \text{ 行第 } l \text{ 列元素为 } A_{k,l}$$

$$\text{计算有 } A_{k,l} = \begin{cases} 4 \cdot \frac{4 + \sum_{i=0}^n (x^i)^2 - 2(x^k)^2}{\left(4 + \sum_{i=0}^n (x^i)^2\right)^2} & l = k \leq m \\ \frac{-8x^k x^j}{\left(4 + \sum_{i=0}^n (x^i)^2\right)^2} & l \neq k \leq m \text{ 为雅可比矩阵的每一项} \\ \frac{16x^l}{\left(4 + \sum_{i=0}^n (x^i)^2\right)^2} & k = m + 1 \end{cases}$$

讲义习题2.3.10

$\mathbb{R}^3$ 中的球坐标系。其中极径 r , 纬度 θ 和经度 φ 如图所示。



(a) 试写出用球坐标 (r, θ, φ) 表达的直角坐标 $(x(r, \theta, \varphi), y(r, \theta, \varphi), z(r, \theta, \varphi))$ 的表达式, 并证明它是可微映射, 求出该映射的 Jacobi 矩阵。该矩阵是否可逆? 为什么?

(b) 记

$$\mathbf{e}_r = \begin{pmatrix} \frac{\partial x}{\partial r} \\ \frac{\partial y}{\partial r} \\ \frac{\partial z}{\partial r} \end{pmatrix}, \quad \mathbf{e}_\theta = \begin{pmatrix} \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial \theta} \end{pmatrix}, \quad \mathbf{e}_\varphi = \begin{pmatrix} \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial \varphi} \\ \frac{\partial z}{\partial \varphi} \end{pmatrix}$$

求它们给出的度量矩阵。

(c) 计算 $(x(r, \theta, \varphi), y(r, \theta, \varphi), z(r, \theta, \varphi))$ 的 Jacobi 矩阵的行列式, 并对结果给出几何解释。

(d) 对可微函数 $f: \mathbb{R}^3 \rightarrow \mathbb{R}$, 求在球坐标系下 f 的梯度 ∇f (用偏导数 $\frac{\partial f}{\partial r}, \frac{\partial f}{\partial \theta}, \frac{\partial f}{\partial \varphi}$ 表示)。

(e) 记 $S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$ 。如果 $f(\theta, \varphi)$ 是 S^2 上的可微函数, 求 ∇f (用偏导数 $\frac{\partial f}{\partial \theta}, \frac{\partial f}{\partial \varphi}$ 表示)。

解: (a) 由几何关系得 $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r \sin \theta \cos \varphi \\ r \sin \theta \sin \varphi \\ r \cos \theta \end{pmatrix}$, 显然该映射可微, 对其求导得到雅可比矩阵

$$\begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} = \begin{pmatrix} \sin \theta \cos \varphi & r \cos \theta \cos \varphi & -r \sin \theta \sin \varphi \\ \sin \theta \sin \varphi & r \cos \theta \sin \varphi & r \sin \theta \cos \varphi \\ \cos \theta & -r \sin \theta & 0 \end{pmatrix} \begin{pmatrix} dr \\ d\theta \\ d\varphi \end{pmatrix} = J \begin{pmatrix} dr \\ d\theta \\ d\varphi \end{pmatrix}$$

(b) 代入计算有 $\mathbf{e}_r = \begin{pmatrix} \sin \theta \cos \varphi \\ \sin \theta \sin \varphi \\ \cos \theta \end{pmatrix}, \mathbf{e}_\theta = \begin{pmatrix} r \cos \theta \cos \varphi \\ r \cos \theta \sin \varphi \\ -r \sin \theta \end{pmatrix}, \mathbf{e}_\varphi = \begin{pmatrix} -r \sin \theta \sin \varphi \\ r \sin \theta \cos \varphi \\ 0 \end{pmatrix}$

$$\langle \mathbf{e}_r, \mathbf{e}_r \rangle = 1, \langle \mathbf{e}_r, \mathbf{e}_\theta \rangle = 0, \langle \mathbf{e}_r, \mathbf{e}_\varphi \rangle = 0, \langle \mathbf{e}_\theta, \mathbf{e}_\theta \rangle = r^2, \langle \mathbf{e}_\theta, \mathbf{e}_\varphi \rangle = 0, \langle \mathbf{e}_\varphi, \mathbf{e}_\varphi \rangle = r^2 \sin^2 \theta$$

则度量矩阵为 $\begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2 \theta \end{pmatrix}$

(c) 由 (a) 中雅可比矩阵计算行列式

$$\begin{aligned} \det J &= r^2 \sin^3 \theta \sin^2 \varphi + r^2 \sin \theta \cos^2 \theta \sin^2 \varphi + r^2 \sin \theta \cos^2 \theta \cos^2 \varphi + r^2 \sin^3 \theta \cos^2 \varphi \\ &= r^2 \sin \theta \sin^2 \varphi + r^2 \sin \theta \cos^2 \varphi = r^2 \sin \theta \end{aligned}$$

该行列式代表三维坐标下微体积元与三维直角坐标系中微体积元之比

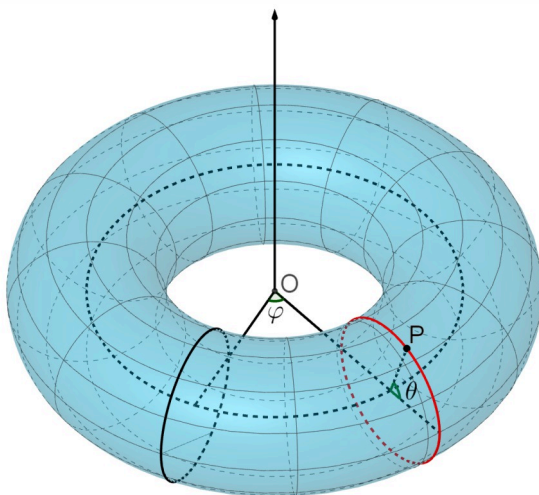
(d) 由梯度表达式计算得

$$\begin{aligned} \nabla f(r, \theta, \varphi) &= (\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\varphi) \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2 \theta \end{pmatrix}^{-1} \begin{pmatrix} \partial_r f \\ \partial_\theta f \\ \partial_\varphi f \end{pmatrix} \\ &= \partial_r f \mathbf{e}_r + \frac{1}{r^2} \partial_\theta f \mathbf{e}_\theta + \frac{1}{r^2 \sin^2 \theta} \partial_\varphi f \mathbf{e}_\varphi = \frac{\partial f}{\partial r} \mathbf{e}_r + \frac{1}{r^2} \frac{\partial f}{\partial \theta} \mathbf{e}_\theta + \frac{1}{r^2 \sin^2 \theta} \frac{\partial f}{\partial \varphi} \mathbf{e}_\varphi \end{aligned}$$

(e) 相当于是固定 $r \equiv 1$, 则 $\nabla f = \frac{\partial f}{\partial \theta} \mathbf{e}_\theta + \frac{1}{\sin^2 \theta} \frac{\partial f}{\partial \varphi} \mathbf{e}_\varphi$

讲义习题2.3.11

环面上, 纬度 θ 和经度 φ 如图所示。设环面轴径为 a , 经线半径为 b , $0 < b < a$, 如图所示。



(a) 试写出用环面坐标 (θ, φ) 表达的直角坐标 $(x(\theta, \varphi), y(\theta, \varphi), z(\theta, \varphi))$ 的表达式, 并证明它是可微映射, 求出该映射的 Jacobi 矩阵。该矩阵是否可逆? 为什么?

(b) 记

$$\mathbf{e}_\theta = \begin{pmatrix} \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial \theta} \end{pmatrix}, \quad \mathbf{e}_\varphi = \begin{pmatrix} \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial \varphi} \\ \frac{\partial z}{\partial \varphi} \end{pmatrix}$$

求它们给出的度量矩阵。

(c) 如果 $f(\theta, \varphi)$ 是环面上的可微函数, 求 ∇f (用偏导数 $\frac{\partial f}{\partial \theta}, \frac{\partial f}{\partial \varphi}$ 表示)。

解: (a) 由几何关系得 $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} (a + b \cos \theta) \cos \varphi \\ (a + b \cos \theta) \sin \varphi \\ b \sin \theta \end{pmatrix}$, 显然其可微, 对其求导有

$$\begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} = \begin{pmatrix} -b \sin \theta \cos \varphi & -(a + b \cos \theta) \sin \varphi \\ -b \sin \theta \sin \varphi & (a + b \cos \theta) \cos \varphi \\ b \cos \theta & 0 \end{pmatrix} \begin{pmatrix} d\theta \\ d\varphi \end{pmatrix} = J \begin{pmatrix} d\theta \\ d\varphi \end{pmatrix}$$

(b) 有 $\mathbf{e}_\theta = \begin{pmatrix} -b \sin \theta \cos \varphi \\ -b \sin \theta \sin \varphi \\ b \cos \theta \end{pmatrix}$, $\mathbf{e}_\varphi = \begin{pmatrix} -(a + b \cos \theta) \sin \varphi \\ (a + b \cos \theta) \cos \varphi \\ 0 \end{pmatrix}$, 计算度量矩阵

$$\langle \mathbf{e}_\theta, \mathbf{e}_\theta \rangle = b^2, \langle \mathbf{e}_\varphi, \mathbf{e}_\varphi \rangle = (a + b \cos \theta)^2, \langle \mathbf{e}_\theta, \mathbf{e}_\varphi \rangle = 0$$

(c) 由梯度表达式 $\nabla f(\theta, \varphi) = (\mathbf{e}_\theta, \mathbf{e}_\varphi) \begin{pmatrix} b^2 & 0 \\ 0 & (a + b \cos \theta)^2 \end{pmatrix}^{-1} \begin{pmatrix} \partial_\theta f \\ \partial_\varphi f \end{pmatrix}$

得到 $\nabla f(\theta, \varphi) = \frac{1}{b^2} \frac{\partial f}{\partial \theta} \mathbf{e}_\theta + \frac{1}{(a + b \cos \theta)^2} \frac{\partial f}{\partial \varphi} \mathbf{e}_\varphi$

讲义习题2.4.5

平面极坐标系下 Laplace 方程为 $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0$ 。

求分离变量形式的解 $u(r, \theta) = X(r)Y(\theta)$ 。

解：代入分离变量形式有 $\Delta u = Y(\theta) \left(\partial_{r,r}^2 X(r) + \frac{1}{r} \partial_r X(r) \right) + \frac{X(r)}{r^2} \partial_{\theta,\theta}^2 Y(\theta) = 0$ 变形为

$$\frac{1}{X(r)} (r^2 \partial_{r,r}^2 X(r) + r \partial_r X(r)) + \frac{1}{Y(\theta)} \partial_{\theta,\theta}^2 Y(\theta) = 0$$

由于前两项与后一项无关联，令 $r^2 \partial_{r,r}^2 X(r) + r \partial_r X(r) = CX(r)$, $\partial_{\theta,\theta}^2 Y(\theta) = -CY(\theta)$

前者为欧拉方程，取 $t = \ln r$ ，化简得 $\frac{\partial^2 X(t)}{\partial t^2} = CX(t)$ ，则分类讨论有

$$\begin{aligned} X(r) &= k_1 \cosh(\sqrt{C} \ln r) + k_2 \sinh(\sqrt{C} \ln r), & C > 0 \\ Y(\theta) &= k_3 \cos(\sqrt{C} \theta) + k_4 \sin(\sqrt{C} \theta); \\ X(r) &= k_1 \cos(\sqrt{-C} \ln r) + k_2 \sin(\sqrt{-C} \ln r), & C < 0 \\ Y(\theta) &= k_3 \cosh(\sqrt{-C} \theta) + k_4 \sinh(\sqrt{-C} \theta); \\ X(r) &= k_1 \ln r + k_2, Y(\theta) = k_3 \theta; & C = 0 \end{aligned}$$

从而合并所有解令 $C > 0$ 有

$$u(r, \theta) = \begin{cases} (C_1 r^C + C_2 r^{-C}) \cos(C\theta + C_3) \\ \cos(C \ln r + C_1) (C_2 e^{C\theta} + C_3 e^{-C\theta}) \\ C_1 (\ln r + C_2) \theta \end{cases}$$

讲义习题2.4.6

求直角坐标系下的 Laplace 方程 $\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$ 在空间球坐标系下表达式。

解：利用 $\Delta = \nabla \cdot \nabla$ ，且由讲义 2.3.10 习题知

$$\begin{aligned} \nabla f(r, \theta, \varphi) &= \frac{\partial f}{\partial r} \mathbf{e}_r + \frac{1}{r^2} \frac{\partial f}{\partial \theta} \mathbf{e}_\theta + \frac{1}{r^2 \sin^2 \theta} \frac{\partial f}{\partial \varphi} \mathbf{e}_\varphi \\ \mathbf{e}_r &= \begin{pmatrix} \sin \theta \cos \varphi \\ \sin \theta \sin \varphi \\ \cos \theta \end{pmatrix}, \mathbf{e}_\theta = \begin{pmatrix} r \cos \theta \cos \varphi \\ r \cos \theta \sin \varphi \\ -r \sin \theta \end{pmatrix}, \mathbf{e}_\varphi = \begin{pmatrix} -r \sin \theta \sin \varphi \\ r \sin \theta \cos \varphi \\ 0 \end{pmatrix} \end{aligned}$$

且不同坐标的偏导有，计算拉普拉斯算子得到

$$\begin{aligned}
\partial_r \mathbf{e}_r &= 0, \partial_r \mathbf{e}_\theta = \frac{\mathbf{e}_\theta}{r}, \partial_r \mathbf{e}_\varphi = \frac{\mathbf{e}_\varphi}{r} \\
\partial_\theta \mathbf{e}_r &= \frac{\mathbf{e}_\theta}{r}, \partial_\theta \mathbf{e}_\theta = -r \mathbf{e}_r, \partial_\theta \mathbf{e}_\varphi = \cot \theta \mathbf{e}_\varphi \\
\partial_\varphi \mathbf{e}_r &= \frac{\mathbf{e}_\varphi}{r}, \partial_\varphi \mathbf{e}_\theta = \cot \theta \mathbf{e}_\varphi, \partial_\varphi \mathbf{e}_\varphi = \begin{bmatrix} -r \sin \theta \cos \varphi \\ -r \sin \theta \sin \varphi \\ 0 \end{bmatrix} \\
\Delta &= \left(\mathbf{e}_r \partial_r + \frac{\mathbf{e}_\theta}{r^2} \partial_\theta + \frac{\mathbf{e}_\varphi}{r^2 \sin^2 \theta} \partial_\varphi \right) \cdot \left(\mathbf{e}_r \partial_r + \frac{\mathbf{e}_\theta}{r^2} \partial_\theta + \frac{\mathbf{e}_\varphi}{r^2 \sin^2 \theta} \partial_\varphi \right) \\
&= \partial_{r,r} + 0 + 0 + \frac{1}{r} \partial_r + \frac{1}{r^2} \partial_{\theta,\theta} + 0 + \frac{1}{r} \partial_r + \frac{\cot \theta}{r^2} \partial_\theta + \frac{1}{r^2 \sin^2 \theta} \partial_{\varphi,\varphi} \\
&= \partial_{r,r} + \frac{2}{r} \partial_r + \frac{1}{r^2} \partial_{\theta,\theta} + \frac{\cot \theta}{r^2} \partial_\theta + \frac{1}{r^2 \sin^2 \theta} \partial_{\varphi,\varphi}
\end{aligned}$$

这与球坐标系下拉普拉斯算子的一般表达式（来源于网络）等价

$$\Delta = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial u}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial u}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 u}{\partial \varphi^2}$$

微积分第四次作业 ($W \times f$)

2022.3.22

教材第一章总复习11

已知偏微分方程 (输运方程) $\begin{cases} \frac{\partial z}{\partial t} = a \frac{\partial z}{\partial x} + b \frac{\partial z}{\partial y}, \\ z(x, y, 0) = z_0(x, y), \end{cases}$ 证明它的解为 $z = z_0(x + at, y + bt)$.

解：显然当 $z = z_0(x + at, y + bt)$ 时， $\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \cdot a + \frac{\partial z}{\partial y} \cdot b$ 满足条件

令 $z(s) = z(x - as, y - bs, t + s)$ ，满足 $\frac{\partial z}{\partial s} = -a \frac{\partial z}{\partial x} - b \frac{\partial z}{\partial y} + \frac{\partial z}{\partial t} = 0$ ，则对每个 x, y, t

都有 $z(x - as, y - bs, t + s) = z(0) = z(x, y, t)$ ，令 $t = -s$ 代入得

$$z(x + at, y + bt, 0) = z_0(x + at, y + bt) = z(x, y, t)$$

则 $z(x, y, t) = z_0(x + at, y + bt)$ 为满足条件的唯一解

教材第一章总复习12(1)

(1) 设函数 $f, g \in C^2$ ，证明： $u(x, y, z, t) = \frac{1}{r}(f(t + r) - g(t - r))$ 满足弦振动方程

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}, \text{ 其中 } r = \sqrt{x^2 + y^2 + z^2};$$

证明：对 $u(x, y, z, t) = \frac{1}{r}(f(t+r) - g(t-r))$ 求偏导有

$$\begin{aligned}\frac{\partial u}{\partial t} &= \frac{f'(t+r) - g'(t-r)}{r}, \quad \frac{\partial^2 u}{\partial t^2} = \frac{f''(t+r) - g''(t-r)}{r} \\ \frac{\partial u}{\partial x} &= \frac{f'(t+r) \cdot x + g'(t-r) \cdot x - (f(t+r) - g(t-r)) \cdot \frac{x}{r}}{x^2 + y^2 + z^2} \\ \frac{\partial^2 u}{\partial x^2} &= \frac{(f'(t+r) + g'(t-r) + f''(t+r) \cdot \frac{x^2}{r} - g''(t-r) \cdot \frac{x^2}{r}) \cdot r^2}{(x^2 + y^2 + z^2)^2} \\ &\quad + \frac{-(f'(t+r) + g'_r(t-r)) \cdot \frac{x^2}{r^2} \cdot r^2 - (f(t+r) - g(t-r)) \cdot \frac{y^2 + z^2}{r^3} \cdot r^2}{(x^2 + y^2 + z^2)^2} \\ &\quad - \frac{(f'(t+r) \cdot x + g'(t-r) \cdot x - (f(t+r) - g(t-r)) \cdot \frac{x}{r}) \cdot 2x}{(x^2 + y^2 + z^2)^2} \\ &= \frac{-(f(t+r) - g(t-r)) \cdot (\frac{-2x^2 + y^2 + z^2}{r}) + (-3x^2 + r^2) \cdot (f'(t+r) + g'(t-r))}{r^4} \\ &\quad + \frac{(f''(t+r) - g''(t-r)) \cdot x^2 r}{r^4}\end{aligned}$$

则对 x, y, z 求和有

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} &= \frac{-(f(t+r) - g(t-r)) \cdot \sum \frac{-2x^2 + y^2 + z^2}{r}}{r^4} \\ &\quad + \frac{(f'(t+r) + g'(t-r)) \cdot \sum (-3x^2 + r^2)}{r^4} + \frac{(f''(t+r) - g''(t-r)) \cdot \sum x^2 r}{r^4} \\ &= 0 + 0 + \frac{f''(t+r) - g''(t-r)}{r} = \frac{\partial^2 u}{\partial t^2}\end{aligned}$$

教材第一章总复习15

设 $f(x, y)$ 是可微函数, 且满足以下条件 $\lim_{x^2+y^2 \rightarrow +\infty} \frac{f(x, y)}{\sqrt{x^2 + y^2}} = +\infty$, 试证明: 对于任意的 $\mathbf{v} = \{v_1, v_2\}$, 都存在点 (x_0, y_0) , 使得 $\text{grad } f(x_0, y_0) = \mathbf{v}$.

证明：对 $\forall v_1, v_2 \in \mathbb{R}, \mathbf{v} = \{v_1, v_2\}$, 定义 $g(x, y) = f(x, y) - (v_1 x + v_2 y)$

则 $\text{grad } f(x_0, y_0) = \mathbf{v} \iff \text{grad } g(x_0, y_0) = 0$, 而 $\frac{|x_0 x + y_0 y|}{\sqrt{x^2 + y^2}} \leq \frac{\sqrt{x_0^2 + y_0^2} \cdot \sqrt{x^2 + y^2}}{\sqrt{x^2 + y^2}}$

后者为一常量 $< +\infty$, 则 $\lim_{x^2+y^2 \rightarrow +\infty} \frac{g(x, y)}{\sqrt{x^2 + y^2}} = +\infty$, 由极限定义知

$$\forall M > 0, \exists N > 0, \forall x^2 + y^2 > N, \frac{g(x, y)}{\sqrt{x^2 + y^2}} > M$$

则任取 M_0 , 令 $\mathbb{S} = \{(x, y) \mid x^2 + y^2 \leq N\}$, 该集合为定义在 $\mathbb{R}^2 \rightarrow \mathbb{R}$ 上的有界闭集

对应值域有最小值 $f(x_1, y_1) = m_{\min}$, 结合上述可知, 该坐标为整个定义域上的最小值

则一定 $\exists x_1, y_1, s. t. \text{grad } g(x_1, y_1) = 0$, 从而原命题成立

讲义2.4.8

求 $\partial^2 \det(A)$ 以及 $\det$ 在 I 处的 2 阶 *Taylor* 展开式。(提示: $AA^{*T} = \det(A)I$ 。)

解: 由 $D \det A(B_1) = \operatorname{tr}(A^{*T} B_1) = \det A \cdot \operatorname{tr}(A^{-1} B_1)$, 继续变化 A 为 B_2 求导有

$D(\operatorname{tr}(A^{-1} B_1))(B_2) = \operatorname{tr}((A + B_2)^{-1} B_1) - \operatorname{tr}(A^{-1} B_1)$ 由可逆运算的一阶“泰勒”展开

$$(A + B_2)^{-1} = (A(I + A^{-1} B_2))^{-1} = (I + A^{-1} B_2)^{-1} A^{-1} = (I - A^{-1} B_2 + o(B_2)) A^{-1}$$

则有 $D(\operatorname{tr}(A^{-1} B_1))(B_2) = -\operatorname{tr}(A^{-1} B_2 A^{-1} B_1)$, 则由链式法则对 $\det A$ 的二阶展开式

$$\partial^2 \det A = \det A \cdot (\operatorname{tr}(A^{-1} B_2) \cdot \operatorname{tr}(A^{-1} B_1) - \operatorname{tr}(A^{-1} B_2 A^{-1} B_1))$$

则 $\det$ 在 I 处的 *Taylor* 展开式有 $1 + \operatorname{tr}(B) + \frac{1}{2}(\operatorname{tr}^2(B) - \operatorname{tr}(B^2)) + o(\operatorname{tr}(B^2))$

讲义2.5.3

求以下函数的极值和最值

(1) $u = \sin x + \sin y + \sin z - \sin(x + y + z) (0 \leq x, y, z \leq \pi)$

(2) $f(\mathbf{x}) = \frac{\langle A\mathbf{x}, \mathbf{x} \rangle}{\langle \mathbf{x}, \mathbf{x} \rangle} (\mathbf{x} \neq \mathbf{0})$, 其中 A 是一个对称矩阵。

(3) $x^y - (x - 1)y + (x - 1)^2 (x > 0, y \in \mathbb{R})$ 。

解: (1) 由对该三元函数求一阶导得

$$\frac{\partial u}{\partial x} = \cos x - \cos(x + y + z), \frac{\partial u}{\partial y} = \cos y - \cos(x + y + z), \frac{\partial u}{\partial z} = \cos z - \cos(x + y + z)$$

当三者均为 0 时有

$$\cos x = \cos y = \cos z = \cos(x + y + z), x, y, z \in [0, \pi]$$

由于 $\cos x$ 函数在 $[0, \pi]$ 区间上是单调的, 从而 $x = y = z$, 代入三倍角公式有

$$\cos x = \cos(3x) = -3\cos x + 4\cos^3 x$$

求二阶导

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} &= -\sin x + \sin(x + y + z), \frac{\partial^2 u}{\partial x \partial y} = \sin(x + y + z), \frac{\partial^2 u}{\partial x \partial z} = \sin(x + y + z) \\ \frac{\partial^2 u}{\partial^2 y} &= -\sin y + \sin(x + y + z), \frac{\partial^2 u}{\partial y \partial z} = \sin(x + y + z), \frac{\partial^2 u}{\partial z^2} = -\sin z + \sin(x + y + z) \end{aligned}$$

解得 $x = 0$ 或 $\frac{\pi}{2}$, 而当 $x = 0$ 为边界点, 不可能为极值点, 代入 $x = \frac{\pi}{2}$ 计算 *Hesse* 矩阵有

$$x = \frac{\pi}{2}, \begin{bmatrix} \frac{\partial^2 u}{\partial x^2} & \frac{\partial^2 u}{\partial x \partial y} & \frac{\partial^2 u}{\partial x \partial z} \\ \frac{\partial^2 u}{\partial y \partial x} & \frac{\partial^2 u}{\partial y^2} & \frac{\partial^2 u}{\partial y \partial z} \\ \frac{\partial^2 u}{\partial z \partial x} & \frac{\partial^2 u}{\partial z \partial y} & \frac{\partial^2 u}{\partial z^2} \end{bmatrix} = \begin{bmatrix} -2 & -1 & -1 \\ -1 & -2 & -1 \\ -1 & -1 & -2 \end{bmatrix}$$

计算该矩阵的特征值

$$\det(\mathbf{H} - \lambda \mathbf{I}) = \begin{vmatrix} -2 - \lambda & -1 & -1 \\ -1 & -2 - \lambda & -1 \\ -1 & -1 & -2 - \lambda \end{vmatrix} = -(\lambda + 1)^2(\lambda + 4), \lambda = -1, -1, -4$$

从而 $\mathbf{H}$ 矩阵负定, 从而 $u(x, y, z) = \sin x + \sin y + \sin z - \sin(x + y + z)$ 在 $(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2})$ 有极大值 4, 同时也是最大值, 对最小值有三角恒等式

$$\sin(x) + \sin(y) + \sin(z) - \sin(x + y + z) = 4 \sin\left(\frac{x + y}{2}\right) \sin\left(\frac{x + z}{2}\right) \sin\left(\frac{y + z}{2}\right)$$

代入取值范围有 $\sin\left(\frac{x + y}{2}\right), \sin\left(\frac{x + z}{2}\right), \sin\left(\frac{y + z}{2}\right) \geq 0$, 则最小值为 0

综上所述原函数有极大值 4, 无极小值, 有最大值 4, 最小值 0

(2) 对该函数求导有

$$\mathbf{d}f(\mathbf{x})\mathbf{v} = \frac{2\mathbf{v}^T A \mathbf{x}}{\mathbf{x}^T \mathbf{x}} - \frac{\mathbf{x}^T A \mathbf{x}}{(\mathbf{x}^T \mathbf{x})^2} 2\mathbf{v}^T \mathbf{x} = \frac{2\mathbf{v}^T}{\mathbf{x}^T \mathbf{x}} [A \mathbf{x} - f(\mathbf{x})\mathbf{x}]$$

$\mathbf{d}f(\mathbf{x})\mathbf{v} = 0 (\forall \mathbf{v})$ 当且仅当 $A \mathbf{x} = f(\mathbf{x})\mathbf{x}$, 即 $\mathbf{x}$ 是 A 的特征向量。此时, $f(\mathbf{x})$ 是相应的特征值。

在 A 的单位特征向量 $\mathbf{x}_0$ 处, $\mathbf{v} = \mu \mathbf{x}_0 + (1 + \mu)\mathbf{v}^\perp$

$$f(\mathbf{x}_0 + \mathbf{v}) = \lambda + (\mathbf{v}^\perp)^T (A - \lambda I) \mathbf{v}^\perp + o(\|\mathbf{v}^\perp\|^2)$$

如果 λ 是 A 的最大特征值 (或最小特征值), 则 $\mathbf{x}_0$ 是 f 的最大值点 (相应地, 最小值点)。如果特征值 λ 既非 A 的最大特征值也非最小特征值, 则 $\mathbf{x}_0$ 不是 f 的极值点。

(3) 对二元函数 $z(x, y) = x^y - (x - 1)y + (x - 1)^2$ 求偏导等于 0 有

$$\begin{aligned} \frac{\partial z}{\partial x} &= yx^{y-1} - y + 2(x - 1) = 0, \frac{\partial z}{\partial y} = x^y \ln x - (x - 1) = 0 \\ \frac{\partial^2 z}{\partial x^2} &= y(y - 1)x^{y-2} + 2, \frac{\partial^2 z}{\partial y^2} = x^y \ln^2 x, \frac{\partial^2 z}{\partial x \partial y} = x^{y-1} + yx^{y-1} \ln x - 1 \end{aligned}$$

固定 y_0 有 $y_0 \leq 0$ 或 $y_0 \leq 1$ 时 $\frac{\partial^2 z}{\partial x^2} \geq 2 > 0$, 则 $\frac{\partial z}{\partial x}$ 随 x 单调递增, 有且仅有 $x = 1$ 该根

若 $0 < y_0 < 1$, 有 $y - 2 < 0$, $\frac{\partial^3 z}{\partial x^3} = y(y - 1)(y - 2)x^{y-3} > 0$, $\frac{\partial^2 z}{\partial x^2}$ 随 x 单调递增

取点 $\frac{\partial^2 z}{\partial x^2} \Big|_{x \rightarrow 0^+} \rightarrow -\infty$, $\frac{\partial^2 z}{\partial x^2} \Big|_{x=1} = 2 > 0$, 由零点存在定理,

$$\exists ! x_0 \in (0, 1), s. t. \frac{\partial^2 z}{\partial x^2} \Big|_{x=x_0} = 0$$

则 $\frac{\partial z}{\partial x}$ 当 $x \in (0, x_0)$ 单调递减, 当 $x \in (x_0, +\infty)$ 单调递增, 又 $\frac{\partial z}{\partial x} \Big|_{x=1} = 0$ 则有

$\frac{\partial z}{\partial x} \Big|_{x=x_0} < 0$ 结合单调性可知, $\frac{\partial z}{\partial x}$ 在 $(0, +\infty)$ 上有且仅有两个零点, 一者 $x_1 \in (0, x_0)$, 一者

为 0, 则 z 关于 x 在区间 $(0, x_1), (x_1, 1), (1, +\infty)$ 上分别单调递增、单调递减、单调递增

但对于 $x_1 \neq 1$ 其在固定 y 的截面上为极大值，而在固定 x 的截面时 $\frac{\partial^2 z}{\partial y^2} = x^y \ln^2 x > 0$ 为极小值

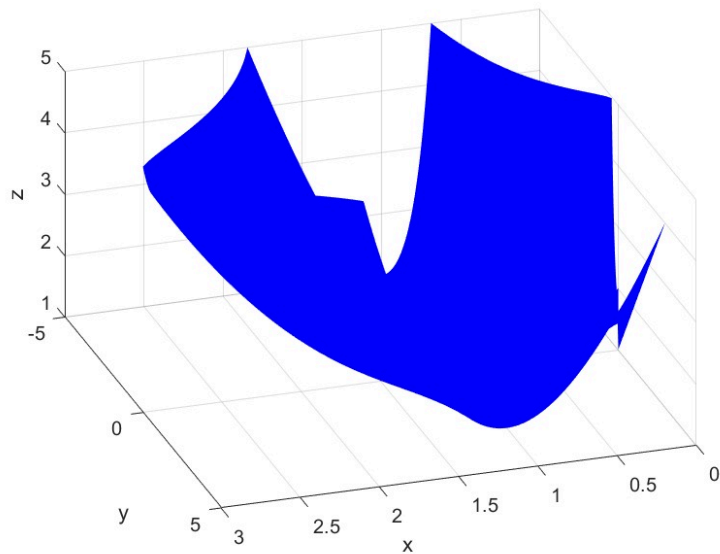
从而对所有的 x_1 均为函数的鞍点，非极值点。而只要极值点坐标 $x = 1$ ，代入 Hesse 矩阵有

$$\mathbf{H} = \begin{bmatrix} \frac{\partial^2 z}{\partial x^2} & \frac{\partial^2 z}{\partial x \partial y} \\ \frac{\partial^2 z}{\partial x \partial y} & \frac{\partial^2 z}{\partial y^2} \end{bmatrix} = \begin{bmatrix} y^2 - y + 2 & 0 \\ 0 & 0 \end{bmatrix}$$

为正定矩阵，从而所有 $x = 1$ 的点（一根直线）都是极小值点，极小值为 1，无极大值

$y \leq 0$ 或 $y \geq 1$ ， $x = 1$ 为最小值， $0 < y < 1$ 时，由单调性 $\min z(x, y) = \min(1, z(x \rightarrow 0, y))$

而 $z(0, y) = 1 - y > 0$ ，从而 $z(x, y)$ 有最小值 1，无最大值，其图像如下



讲义2.5.4

设 γ 是由抛物线 $y = c^2 - x^2$ ($-c \leq x \leq c$) 和 x 轴上的线段组成的简单封闭曲线。求 γ 的内接四边形的最大面积。

解：由于抛物线为凸函数，若四个点均在对称轴一侧，则固定其中三个点，将第四点移动到另一侧显然面积会增加；又若依据对称轴恰好一侧为三个点，一侧为一个点，则按照 y 轴坐标大小排序中间两个点相连总有斜率 k ，若 $k = 0$ ，则可以调整上下任何一个点到以抛物线对称轴为对称中心的对称点使得面积一样；若 $k \neq 0$ ，则总有一侧的点转移至相应对称点后总面积增加。综上，内接四边形面积最大时对称轴分开一定每侧两个点。（假设对称轴上的点划分到右侧）

设横坐标从左往右依次为 $-c \leq x_1 < x_2 < 0 \leq x_3 < x_4 \leq c$ ，固定 x_2, x_3, x_4 有内接四边形面积仅与横坐标为 x_1 的点到经过 x_3, x_4 直线的距离有关，令直线方程 $y = kx + b, k < 0, b > 0$

点到直线距离公式 $d = \frac{|kx_1 + x_1^2 - c^2 + b|}{\sqrt{k^2 + 1}}$ ，其中 $k = -(x_3 + x_4) \in (-2c, 0)$ ， $b = x_3 x_4 + c^2$

代入 $d = \frac{|kx_1 + x_1^2 + x_3 x_4|}{\sqrt{k^2 + 1}}$ ，有 $|kx_1 + x_1^2 + x_3 x_4| \leq |k||c| + c^2 + x_3 x_4$ ，条件 $x_1 = -c$

由对称性可知 $x_1 = -c, x_4 = c$ 。划分为两个三角形相加得到

$$S(x_2, x_3) = \frac{1}{2} \cdot 2c \cdot (c^2 - x_2^2) + \frac{1}{2} \det \begin{vmatrix} c - x_2 & x_2^2 - c^2 \\ x_3 - x_2 & x_2^2 - x_3^2 \end{vmatrix} = c^3 + \frac{1}{2}((x_3 - x_2)c^2 - (x_2^2 + x_3^2)c + (x_3 - x_2)x_2x_3)$$

$$\text{有 } \frac{\partial S}{\partial x_2} = -\frac{1}{2}c^2 - cx_2 + \frac{1}{2}x_3^2 - x_2x_3, \frac{\partial S}{\partial x_3} = \frac{1}{2}c^2 - cx_3 - \frac{1}{2}x_2^2 + x_2x_3$$

当两者等于 0 时, $-2x_2 + x_3 = 2x_3 - x_2 = c$, 则 $x_2 = -\frac{1}{3}c$, $x_3 = \frac{1}{3}c$

$$\text{对应 Hesse 矩阵为 } \mathbf{H} = \begin{bmatrix} \frac{\partial^2 S}{\partial x^2} & \frac{\partial^2 S}{\partial x \partial y} \\ \frac{\partial^2 S}{\partial x \partial y} & \frac{\partial^2 S}{\partial y^2} \end{bmatrix} = \begin{bmatrix} -c - x_3 & x_3 - x_2 \\ x_3 - x_2 & -c + x_2 \end{bmatrix} = \frac{c}{3} \begin{bmatrix} -2 & 2 \\ 2 & -4 \end{bmatrix}$$

为极大值, 且最大值点在极大值处取到, $S_{\max} = \frac{32}{27}c^3$

讲义2.5.5

证明: $f(x, y) = -y + x^2 + xy + x^4 + y^4$ 有唯一极值点, 并判断该极值点的类型。

证明: 对该二元函数求偏导得

$$\frac{\partial f}{\partial x} = 2x + y + 4x^3, \frac{\partial f}{\partial y} = -1 + x + 4y^3$$

则极值点对应 $y = -2x(1 + 2x^2)$, $4y^3 = 1 - x$, 对应 Hesse 矩阵有

$$\mathbf{H} = \begin{bmatrix} \frac{\partial^2 z}{\partial x^2} & \frac{\partial^2 z}{\partial x \partial y} \\ \frac{\partial^2 z}{\partial x \partial y} & \frac{\partial^2 z}{\partial y^2} \end{bmatrix} = \begin{bmatrix} 2 + 8x^2 & 1 \\ 1 & 12y^2 \end{bmatrix}$$

则由于 $2 + 8x^2 > 2 > 0$, 则该对称矩阵 $\mathbf{H}$ 只能为正定矩阵, 其行列式大于等于 0

$$24y^2(1 + 4x^2) \geq 1$$

联立对应方程为 $-32x^3(1 + 2x^2)^3 = 1 - x$, 则解不为 0, $f(x) = 32x^3(1 + 2x^2)^3 - x + 1$

$$f'(x) = 32(3x^2(1 + 2x^2)^3) + x^3 \cdot 12x(1 + 2x^2)^2 - 1 = 96x^2(1 + 2x^2)^2(1 + 6x^2) - 1$$

考虑 $x > 0$ 的情况, $f'(x)$ 在 $x > 0$ 时单调递增, $f(0) = -1$, 且有 $f(\frac{1}{4}) > 6 - 1 > 0$, 由零点存在定理

$\exists x_0 \in (0, \frac{1}{4}), f'(x_0) = 0$, 且有对 $x > 0$, $f(x)$ 在 $(0, x_0)$ 单调递减, 在 $(x_0, +\infty)$

单调递增。而 $f(\frac{1}{4}) = \frac{1}{2} \cdot \frac{9^3}{8} - \frac{3}{4} = -\frac{39}{1024} < 0$, 有 $f(x)$ 在 $(\frac{1}{4}, +\infty)$ 单调递增

且 $f(1) > 1$, $f(x_0) < f(\frac{1}{4}) < 0$, 则在 $x > 0$ 处存在两个零点 x_1, x_2 , 且有 $x_1, x_2 < 1$

放缩不等式有 $48y^2 \cdot \frac{y}{-2x} = 24y^2(2 + 4x^2) \geq 24y^2(1 + 4x^2) \geq 1$, 代入 $4y^3 = 1 - x$

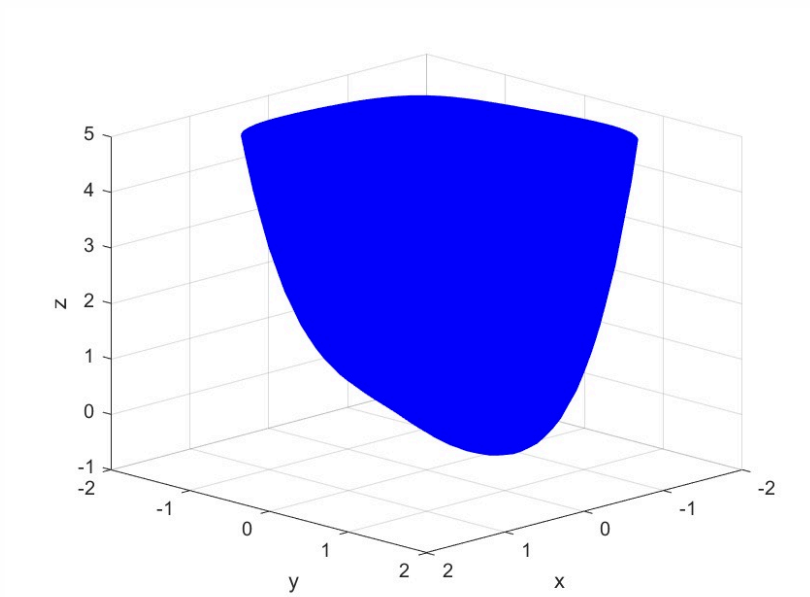
得到 $\frac{6(1-x)}{-x} \geq 1 \implies \frac{1}{x} \leq \frac{5}{6}$, 若 $x > 0$, 则至少需要 $x \geq 1.2$ 才能满足正定矩阵条件, 矛盾

从而 $x < 0$, 由导函数知 $f(x)$ 在 $(-x_0, 0)$ 单调递减, $f(x_0) > f(0) > 1$, 而在 $(-\infty, x_0)$

单调递增，从而由零点唯一性定理，存在唯一负数 $x_3 < 0$ ，使得 $f(x_3) = 0$

$$\text{此时有放缩 } 1 - \frac{1}{x} \geq \frac{1}{3} \implies \frac{12y^3}{-x} = 24y^2 \frac{y}{-2x} \geq 1 \implies 24y^2(1 + 2x^2) \geq 1$$

则满足条件 $24y^2(1 + 4y^2) \geq 24y^2(1 + 2y^2) \geq 1$ ，则只有一个极值点，为极小值点，图像如下



讲义2.8 习题讨论课3.3

求以下函数在指定点的 *Peano* 余项 *Taylor* 展开：

(a) x^y 在 $(x, y) = (1, 0)$ 处展开到二阶；

(b) $\lambda x + e^{-(x^2+y^2)}$ 在它的临界点处展开到二阶。

解：(a) 对多元函数 $z(x, y) = x^y$ ，有一阶导：

$$\frac{\partial z}{\partial x} = y \cdot x^{y-1} = 0, \frac{\partial z}{\partial y} = \ln(x) \cdot x^y = 0$$

二阶导：

$$\frac{\partial^2 z}{\partial x^2} = y(y-1) \cdot x^{y-2} = 0, \frac{\partial^2 z}{\partial y^2} = \ln^2(x) \cdot x^y = 0, \frac{\partial^2 z}{\partial x \partial y} = x^{y-1} + y \ln x \cdot x^{y-1} = 1$$

从而展开有

$$z = 1 + \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)z + \frac{1}{2!} \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^2 z = 1 + (x-1)y$$

即有带 *peano* 余项的 *Taylor* 展开式为 $z = 1 + y(x-1) + o(x^2 + y^2)$

(b) 对多元函数 $z(x, y) = \lambda x + e^{-(x^2+y^2)}$ 有一阶导

$$\frac{\partial z}{\partial x} = \lambda - 2xe^{-(x^2+y^2)}, \frac{\partial z}{\partial y} = -2ye^{-(x^2+y^2)}$$

对于临界点两者取零 $y = 0, \lambda = 2xe^{-x^2} = f(x)$ 有对 $f(x)$ 求导 $f'(x) = (2 - 4x^2)e^{-x^2}$

$x \in (-\infty, -\frac{\sqrt{2}}{2}) \cap (\frac{\sqrt{2}}{2}, +\infty), f'(x) < 0, f(x)$ 单调递减; $x \in (-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$

$f'(x) > 0, f(x)$ 单调递增。当 $x \rightarrow -\infty$ 时, $2xe^{-x^2} \rightarrow 0^-$, $f(-\frac{\sqrt{2}}{2}) = -\sqrt{\frac{2}{e}}$

$f(\frac{\sqrt{2}}{2}) = \sqrt{\frac{2}{e}}$, 结合单调性有 $f(x) \in (-\sqrt{\frac{2}{e}}, \sqrt{\frac{2}{e}})$, 当且仅当 $|\lambda| \leq \sqrt{\frac{2}{e}}$ 时有临界点

而该多元函数的二阶导有

$$\frac{\partial^2 z}{\partial x^2} = (4x^2 - 2)e^{-(x^2+y^2)}, \frac{\partial^2 z}{\partial y^2} = (4y^2 - 2)e^{-(x^2+y^2)}, \frac{\partial^2 z}{\partial x \partial y} = 4xye^{-(x^2+y^2)}$$

记满足 $2xe^{-x^2} = \lambda$ 方程的解为 x_0 (若有多个解, 考虑其中一个解) 对原函数展开有

$$\begin{aligned}\lambda x + e^{-(x^2+y^2)} &= \lambda x_0 + e^{-x_0^2} + \frac{1}{2} \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right)^2 z = \lambda x_0 + e^{-x_0^2} + (2x_0^2 - 1)e^{-x_0^2}(x - x_0) \\ -e^{-x_0^2}y + 0 + o(x^2 + y^2) &= \lambda(x_0 + \frac{1-y}{2x_0} + (2x_0 - \frac{1}{x_0})(x - x_0)) + o(x^2 + y^2)\end{aligned}$$

讲义2.8 习题讨论课3.4

对参数 λ , 讨论函数 $\lambda x + e^{-(x^2+y^2)}$ 的极值情况。

解: 由上一题可知, 当且仅当 $-\sqrt{\frac{2}{e}} \leq \lambda \leq \sqrt{\frac{2}{e}}$ 时有临界点, 而对应 Hesse 矩阵有

$$\mathbf{H} = \begin{bmatrix} \frac{\partial^2 z}{\partial x^2} & \frac{\partial^2 z}{\partial x \partial y} \\ \frac{\partial^2 z}{\partial x \partial y} & \frac{\partial^2 z}{\partial y^2} \end{bmatrix} = 2e^{-(x^2+y^2)} \begin{bmatrix} 2x^2 - 1 & 2xy \\ 2xy & 2y^2 - 1 \end{bmatrix}$$

求解对应特征值

$$\det(\mathbf{H} - \lambda \mathbf{I}) = (2x^2 - 1 - \lambda) \cdot (2y^2 - 1 - \lambda) - 4x^2y^2 = (\lambda + 1)^2 - 2(\lambda + 1)(x^2 + y^2)$$

则特征值为 $\lambda_1 = -1, \lambda_2 = 2x^2 + 2y^2 - 1$, 则若函数存在极值点, 只能 λ_1, λ_2 均小于零

为极大值点, 此时 $2x^2 + 2y^2 \leq 1 \implies x^2 \leq \frac{1}{2}$, 由上一题的单调性可知 $-\sqrt{\frac{2}{e}} \leq \lambda \leq \sqrt{\frac{2}{e}}$

综上所述, $-\sqrt{\frac{2}{e}} \leq \lambda \leq \sqrt{\frac{2}{e}}$ 时有极大值点, 令 $2xe^{-x^2} = \lambda$ 绝对值不超过 $\frac{\sqrt{2}}{2}$ 的解为 x_0 , 该极大值点位于 $(x_0, 0)$, 极大值为 $e^{-x_0^2} + \lambda x_0 = \lambda(x_0 + \frac{1}{2x_0})$, 否则无极值点

微积分第五次作业 ($W \times f$)

2022.3.29

讲义3.1.4

证明 $-y + x^2 + xy + x^4 + y^4 = 0$ 在 $(0, 0)$ 的一个邻域内定义了一个 C^∞ 隐函数 $y = h(x)$, 并求 h 在 $x = 0$ 处的四阶泰勒展开式。

解: 对 $-y + x^2 + xy + x^4 + y^4 = 0$ 令 $x, y \rightarrow 0$ 使用 o 语言有

$$-y + x^2 + o(y) + o(x^2) + o(y) = 0 \implies y = x^2 + o(x^2)$$

令 $y = x^2 + u$ 代入有

$$-u + x^2(x + u) + x^4 + (x^2 + u)^4 = 0 = -u + x^3 + o(u) + o(x^3) + o(x^3) + o(u)$$

则 $u = x^3 + o(x^3)$, 令 $y = x^2 + x^3 + v$ 代入有

$$-x^3 - v + x(x^2 + x^3 + v) + x^4 + (x^2 + x^3 + v)^4 = -v + 2x^4 + o(v) + o(x^4) = 0$$

则 $v = 2x^4 + o(x^4)$, 从而在 $(0, 0)$ 处四阶泰勒展开为 $y = x^2 + x^3 + 2x^4 + o(x^4)$

讲义3.1.5

设 $f: U \times V \rightarrow \mathbb{R}$ 是 C^2 函数, $\mathbf{x}_0$ 是 $f(\cdot, \lambda_0)$ 的非退化临界点, 即 $f(\mathbf{x}, \lambda_0)$ 关于 $\mathbf{x}$ 在 $\mathbf{x}_0$ 处的 *Hesse* 矩阵是可逆的。证明对足够接近 λ_0 的所有 λ , $f(\mathbf{x}, \lambda)$ 关于 $\mathbf{x}$ 在 $\mathbf{x}_0$ 附近有唯一的临界点, 并且该临界点的类型与 $\mathbf{x}_0$ 作为 $f(\cdot, \lambda_0)$ 的临界点的类型是一样的。

证明: 由 $\frac{\partial f}{\partial \mathbf{x}} = \vec{0}$, $\frac{\partial f}{\partial \lambda} = 0$ 有 $\frac{\partial f_1}{\partial \mathbf{x}^i}(\mathbf{x}, \lambda) = 0$, $\frac{\partial f_1}{\partial \lambda}(\mathbf{x}, \lambda) = 0$, 该组隐函数确定了 $\lambda = \lambda(\mathbf{x})$

这是由于 $f(\mathbf{x}_0, \lambda)$ 在 $\mathbf{x} = \mathbf{x}_0$ 处的 *Hesse* 矩阵是可逆的, 从而在 $\mathbf{x}_0$ 附近, 都能找到唯一临界点 λ

又由于 $\det(\mathbf{H})|_{\lambda_0, \mathbf{x}_0} \neq 0$ 且 $\det$ 函数是连续的, 从而总能找到 λ 附近的领域, 使得 $\det \mathbf{H}$ 在该

领域内是保号的, 正定对称矩阵附近仍为正定对称矩阵, 负定也为负定, 从而临界点类型是一样的

讲义3.1.7

设 λ_0 是矩阵 A_0 的一个单重特征值 (即 $p(\lambda_0) = 0, p'(\lambda_0) \neq 0$, 其中 $p(\lambda) = \det(\lambda I - A_0)$), $\mathbf{x}_0$ 是 A_0 对应于 λ_0 的一个单位特征向量。证明存在 $\delta > 0$ 使得对任意矩阵 A , 只要 $\|A - A_0\| < \delta$, A 就有唯一的特征值 $\lambda(A)$ 和相应的单位特征向量 $\mathbf{x}(A)$ 使得 $\lambda(A_0) = \lambda_0, \mathbf{x}(A_0) = \mathbf{x}_0$, 并且 $\lambda(A)$ 和 $\mathbf{x}(A)$ 关于 A 是 C^∞ 的。

证明：由 A 和 λ 由下列隐函数方程确定， $f(A, \lambda) = \det(\lambda I - A) = p_A(\lambda) = 0$

而由题设条件，该方程满足 $\frac{\partial f}{\partial \lambda} \Big|_{\lambda=\lambda_0} = p'(\lambda_0) \neq 0$ ，且在该处只有 n 次求导后的结果是非零的，从而该方程在 A, λ_0 领域所确定的 $\lambda(A)$ 关于 A 只能达到 C^n

而确定 λ 后， $g(A, \lambda, \vec{x}) = g(A, \lambda(A), \vec{x}) = (A - \lambda I)\vec{x} = 0$ ，有 $\frac{\partial g}{\partial \vec{x}^i} = (A - \lambda I)_i$

为 $(A - \lambda I)$ 第 i 列，若其为 0，由行列式性质 $p_A(\lambda I - A) = 0$ ，与题设矛盾，从而均不为 0

则确定了隐函数方程 $\mathbf{x} = \mathbf{x}(A, \lambda(A)) = \mathbf{x}(A)$ ，对高阶导由线性性质，只有对角元上有非零元素，

从而 $\frac{\partial^n f}{\partial \mathbf{x}^n}$ 确定的线性映射都是可逆的，从而 $\mathbf{x}(A)$ 为 C^∞ 的

讲义3.2.1

设

$$f(x_1, x_2) = (x_1 x_2, (1 - x_1)x_2).$$

证明 f 是 $(0, 1) \times (0, 1)$ 到 $\{(y_1, y_2) \mid y_1 > 0, y_2 > 0, y_1 + y_2 < 1\}$ 的微分同胚。

证明：对于变换 $\begin{cases} y_1 = x_1 x_2 \\ y_2 = (1 - x_1)x_2 \end{cases}$ 反解得 $x_1 = \frac{y_1}{y_1 + y_2}, x_2 = y_1 + y_2$

则有 $y_1 + y_2 < 1$ 对应 $x_1 < 1, x_2 < 1$ ，则 f 在 $(0, 1) \mapsto (0, 1)$ 上是双射，满足一一对应，从而

f 微分同胚

讲义3.2.2

记 $\mathcal{S}_n$ 为所有 n 阶对称矩阵组成的线性空间， $\mathcal{P}_n$ 为所有 n 阶对称正定矩阵组成的集合。证明

(a) $\mathcal{P}_n$ 是 $\mathcal{S}_n$ 的开子集；

(b) 对任意 $A \in \mathcal{P}_n$ ，存在唯一的 $A^{1/2} \in \mathcal{P}_n$ 使得 $(A^{1/2})^2 = A$ ，并且 $A^{1/2}$ 关于 A 是 C^∞ 的。

证明：(a) 显然 $\mathcal{S}_n$ 是 $\mathcal{P}_n$ 的真子集，而对于 $A \in \mathcal{P}$ 为对称正定矩阵，由线性代数知识

$$A \text{ 对称正定} \iff \forall \mathbf{x} \in \mathbb{R}^n, \mathbf{x}^T A \mathbf{x} > 0$$

考虑 $f(A) = \mathbf{x}^T A \mathbf{x}$ 关于 A 是连续的，由于对所有 $\mathbf{x}$ 该连续函数 $f(A) > 0$ ，

则总能找到 A 的领域使得所有的 $f(A + E) > 0$ ，即所有 n 阶对称正定矩阵组成的集合为开集

(b) 先证存在性，令 $A = L \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n) L^T, A^{\frac{1}{2}} := L \text{diag}(\sqrt{\lambda_1}, \sqrt{\lambda_2}, \dots, \sqrt{\lambda_n}) L^T$

此时有 $(A^{\frac{1}{2}})^2 = L \text{diag}(\sqrt{\lambda_1}, \sqrt{\lambda_2}, \dots, \sqrt{\lambda_n}) L^T L \text{diag}(\sqrt{\lambda_1}, \sqrt{\lambda_2}, \dots, \sqrt{\lambda_n}) L^T = A$

再证唯一性，若有 $B_1^2 = B_2^2 = A$ ，且 $B_1, B_2 \in \mathcal{P}_n$ ，令 $B_1 = L_1 D_1 L_1^T, B_2 = L_2 D_2 L_2^T$

代入有 $L_1 D_1^2 L_1^T = L_2 D_2^2 L_2^T = A \implies L_2^{-1} L_1 = D_2^2 L_2^T (L_1^T)^{-1} (D_1^{-1})^2$

左侧为上三角矩阵，右侧为下三角矩阵，从而 $L_2^{-1} L_1 = I \implies L_1 = L_2, D_1 = D_2$

即 B 存在且唯一。由该“求根”映射为双射，其逆映射 $A^{\frac{1}{2}} \mapsto A$ 无穷可微

这是因为 $D(A^2)(B) = (A+B)^2 - A^2 = AB + BA + o(\|B\|)$ ，显然可以不断微分下去

由逆映射定理知， $A^{\frac{1}{2}}$ 关于 A 也是 C^∞ 的

讲义3.3.1

设 f, g, h 都是自变量 x, y 的函数, 证明

$$(a) \left(\frac{\partial f}{\partial g} \right)_x = \left(\frac{\partial f}{\partial y} \right)_x / \left(\frac{\partial g}{\partial y} \right)_x ;$$

$$(b) \left(\frac{\partial y}{\partial x} \right)_f = - \left(\frac{\partial f}{\partial x} \right)_y / \left(\frac{\partial f}{\partial y} \right)_x ;$$

$$(c) \left(\frac{\partial f}{\partial x} \right)_g = \left(\frac{\partial f}{\partial x} \right)_y + \left(\frac{\partial f}{\partial y} \right)_x \left(\frac{\partial y}{\partial x} \right)_g .$$

证明: (a) 由所证明式子保持 x 不变, 则有 $df = \frac{\partial f}{\partial y} dy, dg = \frac{\partial g}{\partial y} dy$, 两者相比得

$$\frac{df}{dg}_{x=\text{Const}} = \left(\frac{\partial f}{\partial g} \right)_x = \frac{\left(\frac{\partial f}{\partial y} \right)_x}{\left(\frac{\partial g}{\partial y} \right)_x}$$

(b) 对 $f = f(x, y)$ 全微分有 $df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$, 当 f 不变时计算 $\left(\frac{\partial y}{\partial x} \right)_f$ 得

$$0 = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy \implies \left(\frac{\partial y}{\partial x} \right)_f = - \frac{\left(\frac{\partial f}{\partial x} \right)_y}{\left(\frac{\partial f}{\partial y} \right)_x}$$

(c) 当 g 不变时有, $0 = \frac{\partial g}{\partial x} dx + \frac{\partial g}{\partial y} dy$, 代入 $df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$ 消去 dy 后有

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} \cdot \left(- \frac{\frac{\partial g}{\partial x}}{\frac{\partial g}{\partial y}} dx \right)$$

其中 $dg = \frac{\partial g}{\partial x} dx + \frac{\partial g}{\partial y} dy$, 则固定 g 不变有, $\left(\frac{\partial y}{\partial x} \right)_g = - \frac{\frac{\partial g}{\partial x}}{\frac{\partial g}{\partial y}}$ 代入有

$$\left(\frac{\partial f}{\partial x} \right)_g = \left(\frac{\partial f}{\partial x} \right)_y + \left(\frac{\partial f}{\partial y} \right)_x \left(\frac{\partial y}{\partial x} \right)_g$$

讲义3.3.2

设 $f_1, \dots, f_n; u_1, \dots, u_n$ 都是独立变量 $x_1, \dots, x_n$ 的函数。证明

$$(a) \det \frac{\partial(f_1, \dots, f_n)}{\partial(u_1, \dots, u_n)} = \det \frac{\partial(f_1, \dots, f_n)}{\partial(x_1, \dots, x_n)} / \det \frac{\partial(u_1, \dots, u_n)}{\partial(x_1, \dots, x_n)};$$

$$(b) \left(\frac{\partial f_1}{\partial x_1} \right)_{f_2, \dots, f_n} = \det \frac{\partial(f_1, \dots, f_n)}{\partial(x_1, \dots, x_n)} / \det \frac{\partial(f_2, \dots, f_n)}{\partial(x_2, \dots, x_n)};$$

$$(c) \left(\frac{\partial f_1}{\partial u_1} \right)_{u_2, \dots, u_n} = \det \frac{\partial(f_1, u_2, \dots, u_n)}{\partial(x_1, \dots, x_n)} / \det \frac{\partial(u_1, \dots, u_n)}{\partial(x_1, \dots, x_n)}.$$

证明: (a) 由各组微体积元之比为

$$df_1 df_2 \cdots df_n = \det \left(\frac{\partial(f_1, \dots, f_n)}{\partial(x_1, \dots, x_n)} \right) dx_1 dx_2 \cdots dx_n$$

$$df_1 df_2 \cdots df_n = \det \left(\frac{\partial(f_1, \dots, f_n)}{\partial(u_1, \dots, u_n)} \right) du_1 du_2 \cdots du_n$$

$$du_1 du_2 \cdots du_n = \det \left(\frac{\partial(u_1, \dots, u_n)}{\partial(x_1, \dots, x_n)} \right) dx_1 dx_2 \cdots dx_n$$

$$\text{从而得到 } \det \frac{\partial(f_1, \dots, f_n)}{\partial(u_1, \dots, u_n)} = \frac{df_1 \cdots df_n}{du_1 \cdots du_n} = \det \frac{\partial(f_1, \dots, f_n)}{\partial(x_1, \dots, x_n)} / \det \frac{\partial(u_1, \dots, u_n)}{\partial(x_1, \dots, x_n)}$$

(b) 由固定 $f_2, f_3, \dots, f_n$ 有 $df_2 = df_3 = \cdots = df_n = 0$

$$df_2 = \frac{\partial f_2}{\partial x_1} dx_1 + \cdots + \frac{\partial f_2}{\partial x_n} dx_n, df_3 = \frac{\partial f_3}{\partial x_1} dx_1 + \cdots + \frac{\partial f_3}{\partial x_n} dx_n \cdots, df_n = \cdots$$

$$\text{移项写成矩阵的形式有 } \frac{\partial(f_2, f_3, \dots, f_n)}{\partial(x_2, x_3, \dots, x_n)} \begin{pmatrix} dx_2 \\ dx_3 \\ \vdots \\ dx_n \end{pmatrix} = A \begin{pmatrix} dx_2 \\ dx_3 \\ \vdots \\ dx_n \end{pmatrix} = - \begin{pmatrix} \frac{\partial f_2}{\partial x_2} \\ \frac{\partial f_3}{\partial x_2} \\ \vdots \\ \frac{\partial f_n}{\partial x_2} \end{pmatrix} dx_1 \text{ 代入计算有}$$

$$df_1 = \frac{\partial f_1}{\partial x_1} dx_1 + \cdots + \frac{\partial f_1}{\partial x_n} dx_n = dx_1 \left(\frac{\partial f_1}{\partial x_1} - \begin{pmatrix} \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_1}{\partial x_3} \\ \vdots \\ \frac{\partial f_1}{\partial x_n} \end{pmatrix}^T A^{-1} \begin{pmatrix} \frac{\partial f_2}{\partial x_2} \\ \frac{\partial f_3}{\partial x_2} \\ \vdots \\ \frac{\partial f_n}{\partial x_2} \end{pmatrix} \right) = dx_1 \left(\frac{\partial f_1}{\partial x_1} - \vec{v}_1^T A^{-1} \vec{v}_2 \right)$$

$$\text{由矩阵分块 } B = \det \frac{\partial(f_1, \dots, f_n)}{\partial(x_1, \dots, x_n)} = \det \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \vec{v}_1^T \\ \vec{v}_2 & A \end{pmatrix}$$

$$\frac{\det B}{\det A} = \frac{\det \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \vec{v}_1^T \\ \vec{v}_2 & A \end{pmatrix}}{\det A} = \frac{\det \begin{pmatrix} \frac{\partial f_1}{\partial x_1} - \vec{v}_1^T A^{-1} \vec{v}_2 & \vec{v}_1^T \\ \vec{0} & A \end{pmatrix}}{\det A} = \left(\frac{\partial f_1}{\partial x_1} \right)_{f_2, \dots, f_n}$$

$$\text{即有 } \left(\frac{\partial f_1}{\partial x_1} \right)_{f_2, \dots, f_n} = \det \frac{\partial(f_1, \dots, f_n)}{\partial(x_1, \dots, x_n)} / \det \frac{\partial(f_2, \dots, f_n)}{\partial(x_2, \dots, x_n)}$$

(3) 由固定 $u_2, u_3, \dots, u_n$ 有 $du_2 = du_3 = \dots = du_n = 0$

$$du_2 = \frac{\partial u_2}{\partial x_1} dx_1 + \dots + \frac{\partial u_2}{\partial x_n} dx_n, du_3 = \frac{\partial u_3}{\partial x_1} dx_1 + \dots + \frac{\partial u_3}{\partial x_n} dx_n \dots, du_n = \dots$$

$$\text{移项写成矩阵形式有 } \frac{\partial(u_2, u_3, \dots, u_n)}{\partial(x_2, x_3, \dots, x_n)} \begin{pmatrix} dx_2 \\ dx_3 \\ \vdots \\ dx_n \end{pmatrix} = A \begin{pmatrix} dx_2 \\ dx_3 \\ \vdots \\ dx_n \end{pmatrix} = - \begin{pmatrix} \frac{\partial u_2}{\partial x_2} \\ \frac{\partial u_3}{\partial x_2} \\ \vdots \\ \frac{\partial u_n}{\partial x_2} \end{pmatrix} dx_1, \text{ 代入}$$

$$\text{由上问结论可推知, } du_1 = \frac{\partial u_1}{\partial x_1} dx_1 + \dots + \frac{\partial u_1}{\partial x_n} dx_n = dx_1 \frac{\det \frac{\partial(u_1, \dots, u_n)}{\partial(x_1, \dots, x_n)}}{\det A}$$

$$df_1 = \frac{\partial f_1}{\partial x_1} dx_1 + \dots + \frac{\partial f_1}{\partial x_n} dx_n = dx_1 \left(\frac{\partial f_1}{\partial x_1} - \begin{pmatrix} \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_1}{\partial x_3} \\ \vdots \\ \frac{\partial f_1}{\partial x_n} \end{pmatrix}^T A^{-1} \begin{pmatrix} \frac{\partial u_2}{\partial x_2} \\ \frac{\partial u_3}{\partial x_2} \\ \vdots \\ \frac{\partial u_n}{\partial x_2} \end{pmatrix} \right) = dx_1 \left(\frac{\partial f_1}{\partial x_1} - \vec{v}_1^T A^{-1} \vec{v}_2 \right)$$

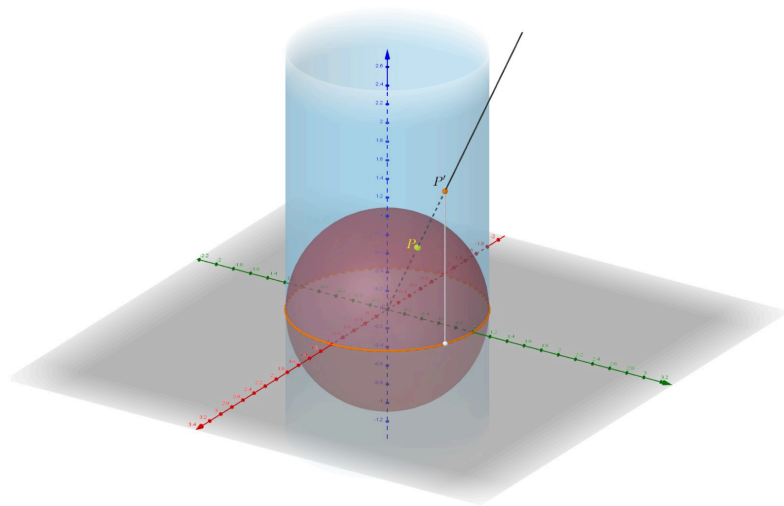
$$\text{由矩阵分块 } C = \det \frac{\partial(f_1, u_2, \dots, u_n)}{\partial(x_1, \dots, x_n)} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \vec{v}_1^T \\ \vec{v}_2 & A \end{pmatrix}$$

$$\frac{\det C}{\det A} = \frac{\det \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \vec{v}_1^T \\ \vec{v}_2 & A \end{pmatrix}}{\det A} = \frac{\det \begin{pmatrix} \frac{\partial f_1}{\partial x_1} - \vec{v}_1^T A^{-1} \vec{v}_2 & \vec{v}_1^T \\ \vec{0} & A \end{pmatrix}}{\det A} = \left(\frac{\partial f_1}{\partial x_1} \right)_{u_2, \dots, u_n}$$

$$\left(\frac{\partial f_1}{\partial u_1} \right)_{u_2, \dots, u_n} = \frac{\det C}{\det A} \cdot \frac{\det A}{\det \frac{\partial(u_1, \dots, u_n)}{\partial(x_1, \dots, x_n)}} = \frac{\det \frac{\partial(f_1, u_2, \dots, u_n)}{\partial(x_1, \dots, x_n)}}{\det \frac{\partial(u_1, \dots, u_n)}{\partial(x_1, \dots, x_n)}}$$

讲义3.4.3

绘制平面地图有很多方法, *Mercator* 投影是其中之一。如图所示, 它把球面上的点 $P(x, y, z)$ 映为从球心出发的射线 OP 与一个圆柱面的交点 $P'(\lambda x, \lambda y, \lambda z)$, 然后在柱面上再做一个变换得到点 $Q(\lambda x, \lambda y, f(\lambda z))$, 该圆柱面与球面相切于赤道。这样除南北两极外, 地球上的点 P 与柱面上的点 Q 一一对应, 把圆柱面沿母线剪开就得到一张地图。求函数 f 的表达式使得 *Mercator* 地图上所画的两条相交道路的夹角与实际道路的夹角大小相等。两条相交曲线的夹角被定义为它们的切线所形成的夹角(锐角或直角)。(提示: 线性变换 $A: \mathbb{R}^n \rightarrow \mathbb{R}^n$ 保角当且仅当它把正交且长度相等的向量映为正交且长度相等的向量, 从而存在常数 $C > 0$ 使得内积 $\langle A\mathbf{u}, A\mathbf{v} \rangle = C\langle \mathbf{u}, \mathbf{v} \rangle, \forall \mathbf{u}, \mathbf{v} \in \mathbb{R}^n$)



解：设球面为 $x^2 + y^2 + z^2 = R^2$ ，对其上一点 (x_0, y_0, z_0) 与圆柱面交线 $(\lambda x_0, \lambda y_0, \lambda z_0)$

满足 $\lambda^2(x_0^2 + y_0^2) = R^2$ 代入变换得 $Q(\frac{Rx_0}{\sqrt{x_0^2 + y_0^2}}, \frac{Ry_0}{\sqrt{x_0^2 + y_0^2}}, f(\frac{Rz_0}{\sqrt{x_0^2 + y_0^2}}))$

考虑经度、纬度恒为定值的两条微元，取纬度 $\varphi \sim \varphi + d\varphi$ 处 $d\theta$ 的正交微圆弧，令其长度相同

$d\theta \cdot R \cos \varphi = R d\varphi \implies d\theta = \frac{d\varphi}{\cos \varphi}$ ，该两条圆弧映射后的长度分别为 $R d\theta$ 和 dz

两者相等有 $R d\theta = dz = R \frac{d\varphi}{\cos \varphi}$ 两边积分得 $z(\varphi) = \int R \frac{d\varphi}{\cos \varphi} = R \ln \left(\frac{1 + \sin \varphi}{\cos \varphi} \right) + C$

代入 $\varphi = 0$ 时 $z = 0$ ，有 $C = 0$ ， $z(\varphi) = R \ln \left(\frac{1 + \sin \varphi}{\cos \varphi} \right)$ ， $\lambda z_0 = \frac{R z_0}{\sqrt{x_0^2 + y_0^2}} = \frac{R z_0}{\sqrt{R^2 - z_0^2}}$

令 $x = \lambda z_0$ 反解得 $z_0 = \frac{Rx}{\sqrt{R^2 + x^2}}$ ， $\frac{1 + \sin \varphi}{\cos \varphi} = \frac{R + z_0}{\sqrt{R^2 - z_0^2}} = \frac{x + \sqrt{x^2 + R^2}}{R}$

从而 $f: x \mapsto R \cdot \operatorname{arcsinh} \frac{x}{R}$ ，为反双曲三角正弦函数

教材1.6.4

设方程 $f(u^2 - x^2, u^2 - y^2, u^2 - z^2) = 0$ 确定了函数 $u = u(x, y, z)$ ，其中 f 可微，证明：

$$\frac{1}{x} \frac{\partial u}{\partial x} + \frac{1}{y} \frac{\partial u}{\partial y} + \frac{1}{z} \frac{\partial u}{\partial z} = \frac{1}{u}.$$

证明：由在方程 $f(u^2 - x^2, u^2 - y^2, u^2 - z^2) = 0$ 下计算得

$$\begin{aligned} \frac{\partial u}{\partial x} &= -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial u}} = -\frac{-2xf'_1}{2u(f'_1 + f'_2 + f'_3)}, \frac{\partial u}{\partial y} = -\frac{\frac{\partial f}{\partial y}}{\frac{\partial f}{\partial u}} = -\frac{-2yf'_1}{2u(f'_1 + f'_2 + f'_3)}, \\ \frac{\partial u}{\partial z} &= -\frac{-2zf'_1}{2u(f'_1 + f'_2 + f'_3)} \rightarrow \frac{1}{x} \frac{\partial u}{\partial x} + \frac{1}{y} \frac{\partial u}{\partial y} + \frac{1}{z} \frac{\partial u}{\partial z} = \frac{f'_1 + f'_2 + f'_3}{u(f'_1 + f'_2 + f'_3)} = \frac{1}{u} \end{aligned}$$

教材1.7.1(2)

求下列曲面在给定点的切平面方程和法线方程.

(2) $z = \arctan \frac{y}{x}$, 点 $P(1, 1, \frac{\pi}{4})$;

解: 方程 $f(x, y, z) = z - \arctan(\frac{y}{x}) = 0$ 在点 $(1, 1, \frac{\pi}{4})$ 处的法向量为

$$\vec{n} = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}) = (\frac{y}{x^2 + y^2}, \frac{-x}{x^2 + y^2}, 1) = (\frac{1}{2}, -\frac{1}{2}, 1)$$

从而切平面方程为 $\frac{1}{2}(x-1) - \frac{1}{2}(y-1) + (z - \frac{\pi}{4}) = 0$, 即 $x - y + 2z = \frac{\pi}{2}$

法线方程 $\frac{x-1}{\frac{1}{2}} = \frac{y-1}{-\frac{1}{2}} = \frac{z - \frac{\pi}{4}}{1}$

教材1.7.4(2)

证明下列各题

(2) 曲面 $\sqrt{x} + \sqrt{y} + \sqrt{z} = \sqrt{a}$ 的任意一点处的切平面在各坐标轴上的截距之和为 a ;

解: 方程 $f(x, y, z) = \sqrt{x} + \sqrt{y} + \sqrt{z} - \sqrt{a} = 0$ 在任意一点 (x_0, y_0, z_0) 处的法向量为

$$\vec{n} = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}) = (\frac{1}{2\sqrt{x}}, \frac{1}{2\sqrt{y}}, \frac{1}{2\sqrt{z}}) = (\frac{1}{2\sqrt{x_0}}, \frac{1}{2\sqrt{y_0}}, \frac{1}{2\sqrt{z_0}})$$

则其切平面方程 $\frac{x-x_0}{2\sqrt{x_0}} + \frac{y-y_0}{2\sqrt{y_0}} + \frac{z-z_0}{2\sqrt{z_0}} = 0$, 则在 x, y, z 轴上的截距分别为

$$x_{inter} = x_0 + (\sqrt{y_0} + \sqrt{z_0})\sqrt{x_0}, y_{inter} = y_0 + (\sqrt{x_0} + \sqrt{z_0})\sqrt{y_0}, z_{inter} = z_0 + (\sqrt{x_0} + \sqrt{y_0})\sqrt{z_0}$$

从而 $x_{inter} + y_{inter} + z_{inter} = x_0 + y_0 + z_0 + 2(\sqrt{x_0 y_0} + \sqrt{x_0 z_0} + \sqrt{y_0 z_0}) = \sqrt{a}^2 = a$

微积分第六次作业 ($W \times f$)

2022.4.5

习题1.9.8

求函数 $u = x^2 + 2y^2 + z^2 - 2xy - 2yz$ 在区域 $x^2 + y^2 + z^2 \leq 4$ 内的最值.

解：由 $u = (x - y)^2 + (y - z)^2 \geq 0$ ，当 $x = y = z = 0$ 时能取到最小值 0

而若最大值在 $x^2 + y^2 + z^2 = r^2 < 4$ 时取到，为 (x_0, y_0, z_0) 则线性扩大 $\frac{2}{r}$ 倍， u 扩大 $\frac{4}{r^2}$ 倍

从而最大值在 $x^2 + y^2 + z^2 = 4$ 时取到，由拉格朗日乘子法

$$f(x, y, z) = (x - y)^2 + (y - z)^2 + \lambda(x^2 + y^2 + z^2 - 4)$$

$$\text{有} \begin{cases} \frac{\partial f}{\partial x} = 2(x - y) + 2\lambda x = 0 \\ \frac{\partial f}{\partial y} = -2(x - y) + 2(y - z) + 2\lambda y = 0, \text{ 则有 } \lambda = \frac{y}{x} - 1 = \frac{y}{z} - 1 = \frac{x + z}{y} - 2 \\ \frac{\partial f}{\partial z} = -2(y - z) + 2\lambda z = 0 \end{cases}$$

若 $y = 0$ ，则 $\lambda = -1, x = -z$ ，代入 $u = x^2 + z^2 = 4$

若 $y \neq 0$ ，得 $x = z, 2x^2 = xy + y^2, x^2 + y^2 + z^2 = 2x^2 + y^2 = 4 = xy + 2y^2, x = \frac{4}{y} - 2y$

有方程 $4 - y^2 = 2(\frac{4}{y} - 2y)^2 = \frac{32}{y^2} - 32 + 8y^2 \implies 9y^4 - 36y^2 + 32 = 0, y^2 = \frac{8}{3}, \frac{4}{3}$

$$u = 2(x - y)^2 = 6(y^2 - x^2) = 6(-3y^2 - \frac{16}{y^2} + 16)$$

代入计算最大为 $6(-6 - 8 + 16) = 12$ 此时 $y = \frac{2\sqrt{6}}{3}, x = -\frac{\sqrt{6}}{3} = z$

综上所述， u 在 $x^2 + y^2 + z^2 \leq 4$ 的最小值为 0，最大值为 12

习题1.9.9(2)

求旋转抛物面 $z = x^2 + y^2$ 与平面 $x + y - z = 1$ 的最短距离；

解：显然从几何上，固定旋转抛物面上的点，该点到平面 $x + y - z = 1$ 的最小距离为垂线段

$$d = \frac{|x_0 + y_0 - z_0 - 1|}{\sqrt{1^2 + 1^2 + 1^2}} = \frac{\sqrt{3}}{3} |x_0 + y_0 - z_0 - 1| = \frac{\sqrt{3}}{3} |x_0 + y_0 - x_0^2 - y_0^2 - 1|$$

而由配方法 $x_0 - x_0^2 = -(x_0 - \frac{1}{2})^2 + \frac{1}{4} \leq \frac{1}{4}$ ，则 $x_0 + y_0 - x_0^2 - y_0^2 - 1 \leq -\frac{1}{2}$

从而取等时对应绝对值最小，则 $d_{\min} = \frac{\sqrt{3}}{6}$

习题1.9.10(1)

水渠的断面为等腰梯形，在渠道表面抹上水泥，当断面面积一定时，梯形的上、下底及腰的长度比例为多少时，所用水泥最省？

解：显然所用水泥质量正比于 $C_{\text{trapesoid}} = l_1 + 2l_{\text{waist}}$

面积为定值 $S = \frac{1}{2}(l_1 + l_2) \cdot h$, 勾股定理 $l_{\text{waist}} = \sqrt{h^2 + (\frac{l_2 - l_1}{2})^2}$

$$C = l_1 + l_2 + \sqrt{4h^2 + (l_2 - l_1)^2} = l_1 + \sqrt{\frac{16S^2}{(l_1 + l_2)^2} + (l_2 - l_1)^2}$$

对 l_2 求导得 $-\frac{32S^2}{(l_1 + l_2)^3} + 2(l_2 - l_1) = 0, 8h^2 = 2(l_2 - l_1)(l_1 + l_2)$

即有 $h^2 = \frac{1}{4}(l_2^2 - l_1^2), l_{\text{waist}} = \frac{1}{2}\sqrt{(l_2 - l_1) \cdot 2l_2}, C = l_1 + \sqrt{2l_2(l_2 - l_1)}$

在约束条件 $16S^2 = (l_2 - l_1)(l_2 + l_1)^3$ 下求 $C(l_1, l_2) = l_1 + \sqrt{2l_2(l_2 - l_1)}$ 的最小值

作拉格朗日函数 $f(l_1, l_2, \lambda) = l_1 + \sqrt{2l_2(l_2 - l_1)} + \lambda(l_2 - l_1)(l_2 + l_1)^3$, 极值点对应

$$\begin{cases} \frac{\partial f}{\partial l_1} = 1 + \frac{-2l_2}{2\sqrt{2l_2(l_2 - l_1)}} + \lambda(l_2 + l_1)^2(2l_2 - 4l_1) = 0 \\ \frac{\partial f}{\partial l_2} = \frac{4l_2 - 2l_1}{2\sqrt{2l_2(l_2 - l_1)}} + \lambda(l_2 + l_1)^2(4l_2 - 2l_1) = 0 \end{cases}$$

令 $\eta = \frac{l_2}{l_1}$ 有 $\frac{\sqrt{2\eta(\eta - 1)} - \eta}{2\eta - 1} = \frac{\eta - 2}{2\eta - 1}$, 解得 $\eta = 2$, 代入 $l_1 : l_2 : l_{\text{waist}} = 1 : 2 : 1$

习题2.2.2(2)

求下列函数的导函数。 $F(y) = \int_{a+y}^{b+y} \frac{\sin yx}{x} dx$;

解: 对 $F(y) = \int_{a+y}^{b+y} \frac{\sin(yx)}{x} dx$, 其对 x 的导函数分为两部分, 分类讨论

$$F'(y) = \frac{\sin(y(b+y))}{b+y} - \frac{\sin(y(a+y))}{a+y} + \int_{a+y}^{b+y} \cos(xy) dx$$

习题2.2.5.(1)

$\int_0^1 \frac{\arctan x}{x} \frac{1}{\sqrt{1-x^2}} dx$ (提示: $\frac{\arctan x}{x} = \int_0^1 \frac{1}{1+x^2y^2} dy$)

解: 由提示变换 $\frac{\arctan x}{x} = \int_0^1 \frac{1}{1+x^2y^2} dy$, 化为二重积分得

$$\int_0^1 \frac{\arctan x}{x} \frac{1}{\sqrt{1-x^2}} dx = \int_0^1 \int_0^1 \frac{1}{1+x^2y^2} \frac{1}{\sqrt{1-x^2}} dx dy$$

代换 $x = \sin \varphi, \varphi \in [0, \frac{\pi}{2}]$, 有 $\frac{1}{1+x^2y^2} \frac{1}{\sqrt{1-x^2}} dx = \frac{d\varphi}{1+y^2 \sin^2 \varphi} = \frac{\cos^2 \varphi \cdot d(\tan \varphi)}{(1+y^2) \sin^2 \varphi + \cos^2 \varphi}$

$$\begin{aligned}\int_0^1 \frac{\arctan x}{x} \frac{1}{\sqrt{1-x^2}} dx &= \int_0^1 \left(\int_0^{\frac{\pi}{2}} \frac{d(\tan \varphi)}{1 + (1+y^2) \tan^2 \varphi} \right) dy = \\ \int_0^1 \left(\frac{\arctan(\sqrt{1+y^2} \tan \varphi)}{\sqrt{1+y^2}} \right) \Big|_0^{\frac{\pi}{2}} dy &= \frac{\pi}{2} \int_0^1 \frac{dy}{\sqrt{1+y^2}} = \\ \frac{\pi}{2} \ln(y + \sqrt{1+y^2}) \Big|_0^1 &= \frac{\pi}{2} \ln(1 + \sqrt{2})\end{aligned}$$

习题2.3.1

计算下列积分.

$$(1) \int_0^{+\infty} \frac{e^{-ax^2} - e^{-bx^2}}{x} dx (a, b > 0);$$

$$(3) \int_0^{+\infty} \frac{\cos ax - \cos bx}{x^2} dx (a, b > 0)$$

解: (1) 解: $\int_0^{+\infty} \frac{e^{-ax^2} - e^{-bx^2}}{x} dx = \int_0^{+\infty} \frac{1}{x} dx \int_a^b x^2 e^{-x^2 y} dy = \int_0^{+\infty} \int_a^b x e^{-x^2 y} dx dy$

$$\int_0^{+\infty} x e^{-x^2 y} dx = \frac{1}{2} \int_0^{+\infty} e^{-x^2 y} d(x^2) = -\frac{1}{2y} \cdot e^{-x^2 y} \Big|_0^{+\infty} = \frac{1}{2y}$$

$$\int_0^{+\infty} \frac{e^{-ax^2} - e^{-bx^2}}{x} dx = \int_a^b \frac{1}{2y} dy = \frac{1}{2} \ln\left(\frac{b}{a}\right)$$

$$(3) \int_0^{+\infty} \frac{\cos ax - \cos bx}{x^2} dx = - \int_0^{+\infty} \frac{1}{x^2} dx \int_b^a x \sin(xy) dy = \int_0^{+\infty} \int_a^b \frac{\sin(xy)}{x} dx dy$$

$$\text{set } t = xy \in [0, +\infty), \int_0^{+\infty} \frac{\sin(xy)}{x} dx = \int_0^{+\infty} \frac{y \sin t}{t} d\left(\frac{t}{y}\right) = \int_0^{+\infty} \frac{\sin t}{t} dt = \frac{\pi}{2}$$

代入得 $\int_0^{+\infty} \frac{\cos ax - \cos bx}{x^2} dx = \int_a^b \frac{\pi}{2} dy = \frac{\pi}{2}(b-a)$

习题2.1.4

讨论下列积分在所给区间上的一致收敛性.

$$(2) \int_{-\infty}^{+\infty} \frac{\cos yx}{1+x^2} dx (-\infty < y < +\infty);$$

$$(9) \int_1^{+\infty} x^{1-y} dx (0 < y < +\infty);$$

$$(10) \int_0^{+\infty} \frac{\sin x^2}{x^p} dx (0 \leq p < +\infty).$$

解: (2) 由 $\left| \frac{\cos yx}{1+x^2} \right| \leq \frac{1}{1+x^2}$, 且有 $\int_{-\infty}^{+\infty} \frac{1}{1+x^2} dx = \arctan x \Big|_{-\infty}^{+\infty} = \pi$

对于 $\int_{-\infty}^{+\infty} \left| \frac{\cos yx}{1+x^2} \right| dx = 2 \int_0^{+\infty} \left| \frac{\cos yx}{1+x^2} \right| dx$ 收敛, 由柯西收敛准则

$$\forall \epsilon > 0, \exists A > 0, s. t. \forall A_1 > A_2 > A, \int_{A_1}^{A_2} \left| \frac{\cos yx}{1+x^2} \right| dx < \epsilon$$

放缩绝对值有 $\forall \epsilon > 0, \exists A > 0, s. t. \forall A_1 > A_2 > A, \left| \int_{A_1}^{A_2} \frac{\cos yx}{1+x^2} dx \right| \leq \int_{A_1}^{A_2} \left| \frac{\cos yx}{1+x^2} \right| dx < \epsilon$

由柯西收敛准则得原广义含参积分收敛

(9) 当 $0 < y \leq 2$ 时 $x^{1-y} \geq x^{-1} = \frac{1}{x}$, 则

$$\exists \epsilon_0 = \ln 2, \forall A > 1, s. t. \exists A_2 > 2A_1 > A, \int_{A_1}^{A_2} x^{1-y} dx > \int_{A_1}^{A_2} \frac{1}{x} dx = \ln x \Big|_{A_1}^{A_2} > \epsilon_0$$

此时广义含参积分不收敛; 若当 $2 < y$ 时, 直接计算有 $\frac{1}{2-y} \cdot (0-1) = \frac{1}{y-2}$ 收敛

$$(10) \text{ 代换 } t = x^2 \in [0, \infty), \int_0^{+\infty} \frac{\sin x^2}{x^p} dx = \int_0^{+\infty} \frac{\sin t}{t^{\frac{p}{2}}} d(\sqrt{t}) = \int_0^{+\infty} \frac{\sin t}{t^{\frac{p+1}{2}}} dt$$

$$\text{其中 } \alpha = \frac{p+1}{2} \geq \frac{1}{2}, \int_1^{+\infty} \frac{\sin t}{t^\alpha} dt = \int_1^{+\infty} \frac{-d(\cos t)}{t^\alpha} = -\frac{\cos t}{t^\alpha} \Big|_1^{+\infty} - \int_1^{+\infty} \frac{\alpha dt}{t^{\alpha+1}}$$

$$= \cos 1 - \alpha \int_1^{+\infty} \frac{\cos t dt}{t^{\alpha+1}}, \text{ 而放缩 } \int_1^{+\infty} \left| \frac{\cos t}{t^{\alpha+1}} \right| dt \leq \int_1^{+\infty} \frac{dt}{t^{\alpha+1}} = \frac{1}{\alpha}, \text{ 后者绝对收敛}$$

从而 $\int_1^{+\infty} \frac{\sin t}{t^\alpha} dt$ 一致收敛, 只需考虑 $\int_0^1 \frac{\sin t}{t^\alpha} dt$, 这与 $\lim_{t \rightarrow 0} \frac{\sin t}{t^\alpha}$ 的收敛性相同

当且仅当 $\alpha = \frac{p+1}{2} \leq 1$ 即 $0 \leq p \leq 1$ 时原广义含参积分收敛, $1 < p$ 时不收敛

微积分第七次作业 ($W \times f$)

2022.4.24

习题3.2.3

比较下列各组积分值的大小.

$$(2) \iint_D \ln(x+y) dx dy, \iint_D xy dx dy, D \text{ 由直线 } x=0, y=0, x+y=\frac{1}{2}, x+y=1$$

围成.

解: 去掉边界积分结果不变, 由于 $x+y < 1$, 则 $\ln(x+y) < 0$ 而 $xy > 0$, 且积分区域 $x, y > 0$

$$\text{从而 } \ln(x+y) dx dy < xy dx dy \text{ 则 } \iint_D \ln(x+y) dx dy < \iint_D xy dx dy$$

习题3.3.6

计算下列二重积分.

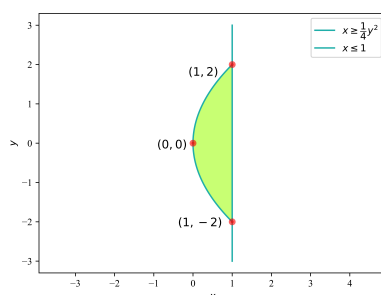
(1) $\iint_D xy^2 \, dx \, dy, D = \{(x, y) \mid 4x \geq y^2, x \leq 1\};$

(2) $\iint_D \frac{1}{\sqrt{2a-x}} \, dx \, dy, D = \{(x, y) \mid (x-a)^2 + (y-a)^2 \leq 1, 0 \leq x, y \leq a\};$

(4) $\iint_D x \cos(xy) \, dx \, dy, D = \{(x, y) \mid x^2 + y^2 \leq R^2\};$

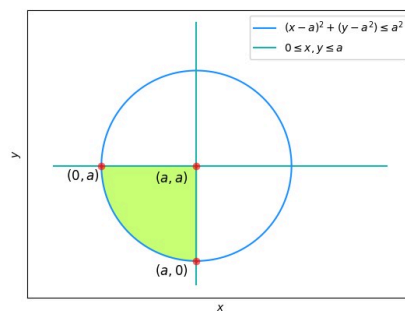
(9) $\iint_D y^2 \, dx \, dy, D$ 由 $\begin{cases} x = a(t - \sin t), \\ y = a(1 - \cos t), \end{cases} \quad 0 \leq t \leq 2\pi$ 以及 x 轴围成;

解: (1) 积分区域图像如右 $\begin{cases} x \geq \frac{1}{4}y^2 \\ x \leq 1 \end{cases}$



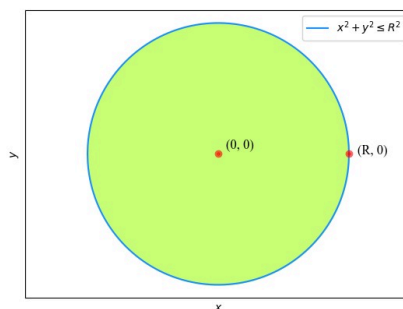
$$\iint_D xy^2 \, dx \, dy = \int_0^1 dx \int_{-2\sqrt{x}}^{2\sqrt{x}} xy^2 \, dy = \int_0^1 x \cdot \frac{2}{3} (2\sqrt{x})^3 \, dx = \frac{16}{3} \int_0^1 x^{\frac{5}{2}} \, dx = \frac{32}{21}$$

(2) 积分区域 $\begin{cases} (x-a)^2 + (y-a)^2 \leq a^2 \\ 0 \leq x, y \leq a \end{cases}$ 图像如右



$$\begin{aligned} \iint_D \frac{1}{\sqrt{2a-x}} \, dx \, dy &= \int_0^a dx \int_{a-\sqrt{x(2a-x)}}^a \frac{1}{\sqrt{2a-x}} \, dy \\ &= \int_0^a \sqrt{x(2a-x)} \cdot \frac{1}{\sqrt{2a-x}} \, dx = \frac{2}{3} a^{\frac{3}{2}} \end{aligned}$$

(4) 积分区域 $D: x^2 + y^2 \leq R^2$ 图像如右

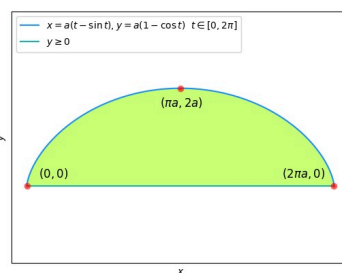


$$\text{拆成两部分 } \iint_D x \cos(xy) dx dy = \iint_{D \cap \{x \leq 0\}} x \cos(xy) dx dy + \iint_{D \cap \{x > 0\}} x \cos(xy) dx dy$$

对 $f(x, y) = x \cos(xy)$ 有 $f(x, y) = -f(-x, y)$ 为关于 x 的奇函数, 则 $\iint_D x \cos(xy) dx dy = 0$

$$\begin{aligned} \iint_D |\cos(x+y)| dx dy &= \iint_{D'} |\cos(x+y)| \frac{D(x, y)}{D(v, u)} dv du = \iint_D |\cos(x+y)| \left| \begin{array}{cc} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{array} \right| du dv \\ &= \frac{1}{2} \iint_D |\cos(x+y)| dv du = \frac{1}{2} \cdot 4 \int_0^\pi |\cos u| du \int_0^u dv = 2 \cdot 2 \int_0^{\frac{\pi}{2}} u d(\sin u) = 2\pi - 4 \end{aligned}$$

(9) 解: 滚轮线 $\begin{cases} x = a(t - \sin t) \\ y = a(1 - \cos t) \end{cases} t \in [0, 2\pi], D: \begin{cases} y \leq f(x) \\ 0 \leq y \end{cases}$ 图像



$$\iint_D y^2 dx dy = \int_0^{2\pi a} dx \int_0^{f(x)} y^2 dy = \frac{1}{3} \int_0^{2\pi a} f(x)^3 dx, \text{ 换成参数 } t \text{ 积分得}$$

$$\begin{aligned} &\frac{1}{3} \int_0^{2\pi} a^4 (1 - \cos t)^3 d(t - \sin t) = \frac{a^4}{3} \int_0^{2\pi} (1 - \cos t)^4 dt = \frac{a^4}{3} \int_{-\pi}^{\pi} (1 + \cos x)^4 dx = \\ &\frac{a^4}{3} \int_{-\pi}^{\pi} (1 + 4 \cos x + 6 \cos^2 x + 4 \cos^3 x + \cos^4 x) dx = \frac{a^4}{3} \int_{-\pi}^{\pi} (1 + 6 \cos^2 x + \cos^4 x) dx \\ &= \frac{a^4}{3} (2\pi + \int_{-\pi}^{\pi} 3(1 + \cos 2x) dx + \int_{-\pi}^{\pi} \frac{(1 + \cos 2x)^2}{4} dx) = \frac{a^4}{3} (8\pi + \int_{-\pi}^{\pi} \frac{1 + \frac{1}{2}}{4} dx + 0) \\ &= \frac{a^4}{3} (8\pi + \frac{3\pi}{4}) = \frac{35\pi}{12} a^4 \quad (\text{答案有误, 分母缺少3}) \end{aligned}$$

习题3.3.12

(3) $\iint_D (x+y) dx dy$, D 是由 $x^2 + y^2 = x + y$ 围成的平面区域

解: 该区域为一个圆 $(x - \frac{1}{2})^2 + (y - \frac{1}{2})^2 = (\frac{\sqrt{2}}{2})^2$ 换元 $\begin{cases} x = \frac{1}{2} + r \cos \theta \\ y = \frac{1}{2} + r \sin \theta \end{cases}$

得到 $dx dy = r d\theta dr$ 计算有 $\iint_D (x+y) dx dy = \iint_D (1 + r \cos \theta + r \sin \theta) r d\theta dr$

$$\begin{aligned} \iint_D (1 + r \cos \theta + r \sin \theta) r d\theta dr &= \int_0^{\frac{\sqrt{2}}{2}} dr \int_0^{2\pi} (1 + r \cos \theta + r \sin \theta) \cdot r d\theta \\ &= \int_0^{\frac{\sqrt{2}}{2}} dr (2\pi + 0 + 0) \cdot r = \pi r^2 \Big|_0^{\frac{\sqrt{2}}{2}} = \frac{\pi}{2} \end{aligned}$$

习题3.3.16

设函数 $f(t)$ 连续, 证明:

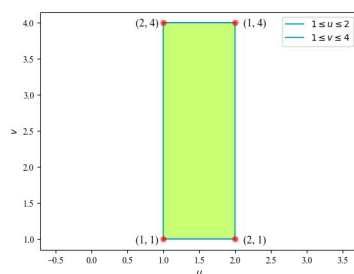
$$(2) \iint_{|x|+|y|\leq 1} f(xy) dx dy = \ln 2 \int_1^2 f(t) dt, D \text{ 是由 } xy=1, xy=2, y=x, y=4x \text{ 所围成}$$

第一象限的区域.

解: 代换 $xy = u, \frac{y}{x} = v$, 逆变换 $x = \sqrt{\frac{u}{v}}, y = \sqrt{uv}$, 则雅可比行列式

$$\frac{D(x, y)}{D(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2\sqrt{uv}} & -\frac{1}{2}\sqrt{\frac{u}{v^3}} \\ \frac{1}{2}\sqrt{\frac{v}{u}} & \frac{1}{2}\sqrt{\frac{u}{v}} \end{vmatrix} = \frac{1}{2v}$$

原区域 $D: \begin{cases} 1 \leq xy \leq 2 \\ 1 \leq \frac{y}{x} \leq 4 \end{cases}$ 变换后对应 $D': \begin{cases} 1 \leq u \leq 2 \\ 1 \leq v \leq 4 \end{cases}$ 图像如右



$$\iint_D f(xy) dx dy = \iint_{D'} f(u) \cdot \frac{1}{2v} du dv = \int_1^2 f(u) du \int_1^4 \frac{dv}{2v} = \ln 2 \int_1^2 f(t) dt$$

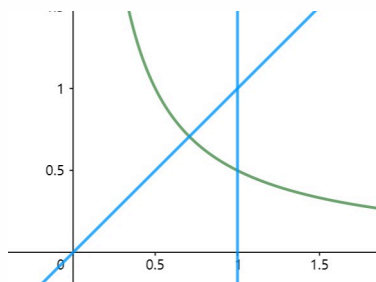
习题3.4.5

计算下列三重积分的值.

(1) $\iiint_{\Omega} xy^2 z^3 dx dy dz$, Ω 是由马鞍面 $z = xy$ 与平面 $y = x, x = 1, z = 0$ 所围成空间区域;

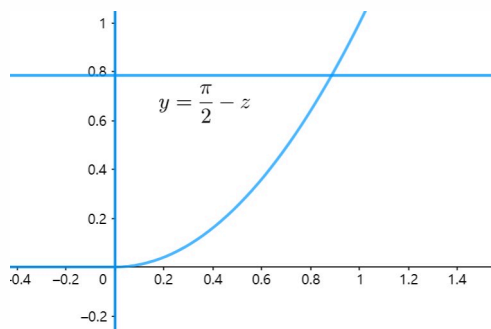
(3) $\iiint_{\Omega} x \cos(y+z) dx dy dz$, Ω 是由曲面 $x = \sqrt{y}$ 与平面 $x = 0, z = 0, y+z = \frac{\pi}{2}$ 围成区域;

解: (1) 从 $z = 0$ 连续变化到 $z = 1$, 图中画出 $z = \frac{1}{2}$ 的情况



对每个 z 二重积分得到

$$\begin{aligned}
\iiint_{\Omega} xy^2z^3 \, dx \, dy \, dz &= \int_0^1 dz \int_{\sqrt{z}}^1 dx \int_{\frac{z}{x}}^x xy^2z^3 dy = \int_0^1 z^3 dz \int_{\sqrt{z}}^1 x dx \frac{y^3}{3} \Big|_{\frac{z}{x}}^x \\
&= \frac{1}{3} \int_0^1 z^3 dz \int_{\sqrt{z}}^1 x dx (x^3 - \frac{z^3}{x^3}) = \frac{1}{3} \int_0^1 z^3 dz (\frac{1}{5} x^5 + \frac{z^3}{x}) \Big|_{\sqrt{z}}^1 \\
&= \frac{1}{3} \int_0^1 z^3 (\frac{1}{5} - \frac{1}{5} z^{\frac{5}{2}} + z^3 - z^{\frac{5}{2}}) dz = \frac{1}{3} (\frac{1}{20} - \frac{6}{5} \cdot \frac{2}{13} + \frac{1}{7}) = \frac{1}{364}
\end{aligned}$$



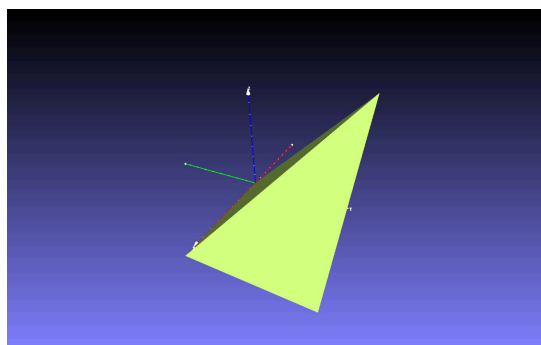
(3) 对 $z = 0$ 连续变化到 $z = \frac{\pi}{2}$, 图像如右

$$\begin{aligned}
\iiint_{\Omega} x \cos(y+z) \, dx \, dy \, dz &= \int_0^{\frac{\pi}{2}} dz \int_0^{\frac{\pi}{2}-z} dy \int_0^{\sqrt{y}} x \cos(y+z) \, dx \\
&= \frac{1}{2} \int_0^{\frac{\pi}{2}} dz \int_0^{\frac{\pi}{2}-z} dy \cdot y \cos(y+z) = \frac{1}{2} \int_0^{\frac{\pi}{2}} dz (y \sin(y+z) \Big|_0^{\frac{\pi}{2}-z} - \int_0^{\frac{\pi}{2}-z} \sin(y+z) dy) \\
&= \frac{1}{2} \int_0^{\frac{\pi}{2}} dz ((\frac{\pi}{2} - z) + \cos(y+z) \Big|_0^{\frac{\pi}{2}-z}) = \frac{1}{2} \int_0^{\frac{\pi}{2}} ((\frac{\pi}{2} - z) - \cos z) dz \\
&= \frac{1}{2} (\frac{\pi^2}{4} - \frac{\pi^2}{8}) - \frac{1}{2} = \frac{\pi^2 - 8}{16}
\end{aligned}$$

习题3.4.6

计算累次积分 $I = \int_0^1 dx \int_0^x dy \int_0^y \frac{\cos z}{(1-z)^2} dz$ 的值.

解：三维图形为三棱锥，图像如右（使用 meshlab 绘图）



在 $z = z_0$ 切面上由面积相似比等于高之比的平方 $\eta = (1-z)^2$, 从而有

$$\int_0^1 dx \int_0^x dy \int_0^y \frac{\cos z}{(1-z)^2} dz = \int_0^1 \frac{\cos z}{(1-z)^2} \cdot S dz = \int_0^1 \frac{\cos z}{(1-z)^2} \frac{1}{2} (1-z)^2 dz = \frac{\sin 1}{2}$$

习题3.4.7

计算下列三重积分的值.

$$(4) \iiint_{\Omega} x e^{\frac{x^2+y^2+z^2}{a^2}} dx dy dz, \Omega = \{(x, y, z) \mid x^2 + y^2 + z^2 \leq a^2, x, y, z \geq 0\}$$

解：使用球坐标 $(x, y, z) = (r, \theta, \varphi)$ 有 $dx dy dz = r^2 \sin \theta dr d\theta d\varphi$ 计算积分有

$$\begin{aligned} \iiint_{\Omega} x e^{\frac{x^2+y^2+z^2}{a^2}} dx dy dz &= \iiint_{\Omega} r \sin \theta \sin \varphi e^{\frac{r^2}{a^2}} r^2 \sin \theta dr d\theta d\varphi \\ &= \int_0^{\frac{\pi}{2}} \frac{1 - \cos 2\theta}{2} d\theta \int_0^a r^3 e^{\frac{r^2}{a^2}} dr = \int_0^{a^2} \frac{\pi}{8} t e^{\frac{t}{a^2}} dt = \frac{\pi}{8} (a^2 t - a^4) e^{\frac{t}{a^2}} \Big|_0^{a^2} = \frac{\pi a^4}{8} \end{aligned}$$

习题3.4.10

$f(t)$ 在 $(-\infty, +\infty)$ 连续, $f(t) = 3 \iiint_{x^2+y^2+z^2 \leq t^2} f(\sqrt{x^2+y^2+z^2}) dx dy dz + |t^3|$, 求 $f(t)$

解：积分区域为球体 $x^2 + y^2 + z^2 \leq t^2$, 换取球坐标 $(x, y, z) \rightarrow (\rho, \theta, \varphi)$, 有体积元变换

$$\begin{aligned} dx dy dz &= \rho^2 \sin \theta d\rho d\theta d\varphi \\ f(t) &= 3 \int_0^{2\pi} \int_0^{\pi} \int_0^t f(\rho) \rho^2 \sin \theta d\rho d\theta d\varphi + |t^3| \\ &= 3 \cdot 2 \cdot 2\pi \int_0^t f(\rho) \rho^2 d\rho + |t^3| = 12\pi \int_0^t f(\rho) \rho^2 d\rho + |t^3| \end{aligned}$$

两边对 t 求导得

$$f'(t) = 12\pi t^2 f(t) + \text{sign}(t) \cdot 3t^2, \text{ where } \text{sign}(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$$

其特解与 t 的符号有关 $\begin{cases} t > 0, f_0 = -\frac{1}{4\pi} \\ t < 0, f_0 = \frac{1}{4\pi} \end{cases}$, 其通解满足方程

$$\frac{df}{dt} = 12\pi t^2 f \implies \int \frac{df}{f} = \int 12\pi t^2 dt, \ln f = 4\pi t^3 + C \therefore f(t) = \begin{cases} C_1 e^{4\pi t^3} - \frac{1}{4\pi}, & t > 0 \\ C_2 e^{4\pi t^3} + \frac{1}{4\pi}, & t < 0 \end{cases}$$

由 $f(t)$ 在 $(-\infty, +\infty)$ 连续得 $C_1 - \frac{1}{4\pi} = C_2 + \frac{1}{4\pi}$, 有 $C_1 = C_2 + \frac{1}{2\pi}$, 对 $t > 0$ 代入原积分

$$\begin{aligned} f(t) &= C_1 e^{4\pi t^3} - \frac{1}{4\pi} = 12\pi \int_0^t (C_1 e^{4\pi \rho^3} - \frac{1}{4\pi}) \rho^2 d\rho + t^3 \\ &= \int_0^t C_1 e^{4\pi \rho^3} d(4\pi \rho^3) + 0 = C_1 (e^{4\pi t^3} - 1) \end{aligned}$$

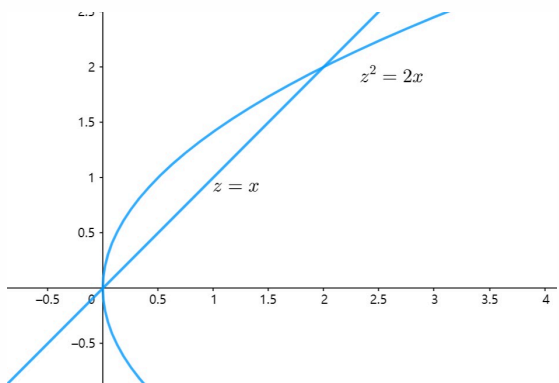
比对得 $C_1 = \frac{1}{4\pi}$, $C_2 = -\frac{1}{4\pi}$, 从而 $f(t) = \frac{|e^{4\pi t^3} - 1|}{4\pi}$

习题3.5.1

求下列曲面的面积.

(2) 锥面 $z = \sqrt{x^2 + y^2}$ 在柱面 $z^2 = 2x$ 内的部分;

解: $z = \sqrt{x^2 + y^2}$ 和 $z^2 = 2x$ 在 $y = 0$ 处截图如下



取柱坐标 $(x, y, z) \mapsto (\rho, \theta, z)$ 则锥面 $z = \rho, z^2 = 2\rho \cos \theta$ 其要求 $\rho \leq 2 \cos \theta$ 为圆

$$\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} = \sqrt{2} \text{ 则 } S = \iint_D \sqrt{2} dx dy = \sqrt{2} \cdot S_{\text{圆}} = \sqrt{2}\pi$$

习题3.5.2

求下列曲面所包围的均匀物体的质心.

(3) $z = 4 - x^2 (x \geq 0), x = 0, y = 0, z = 0, y = 6$.

解: 由对称性该物体的质心 y 坐标为 3, 且质心的 x, z 坐标与平面内抛物三角形相同

$$x_c = \frac{\int_0^2 x(4 - x^2) dx}{\int_0^2 (4 - x^2) dx} = \frac{4}{\frac{16}{3}} = \frac{3}{4}, z_c = \frac{\int_0^2 \frac{4 - x^2}{2} (4 - x^2) dx}{\int_0^2 (4 - x^2) dx} = \frac{\frac{128}{15}}{\frac{16}{3}} = \frac{8}{5}$$

则质心坐标为 $(\frac{3}{4}, 3, \frac{8}{5})$

习题4.2.1

计算下列曲线积分.

(4) $\int_L (x^{\frac{4}{3}} + y^{\frac{4}{3}}) dl$, 其中 L 为星形线 $\begin{cases} x = a \cos^3 t, \\ y = a \sin^3 t, \end{cases} 0 \leq t \leq 2\pi$.

解: 由曲线微元 $dl = \sqrt{dx^2 + dy^2} = 3a |\cos t \sin t|$ 代入有

$$\begin{aligned}
& \int_L (x^{\frac{4}{3}} + y^{\frac{4}{3}}) 3a |\cos t \sin t| dt = 3 \int_0^{2\pi} a^{\frac{7}{3}} (\cos^4 t + \sin^4 t) |\cos t \sin t| dt \\
& = 12a^{\frac{7}{3}} \int_0^{\frac{\pi}{2}} (1 - \frac{1}{4} \sin^2 2t) \frac{1}{2} \sin 2t dt = 12a^{\frac{7}{3}} (\frac{1}{2} - \int_0^{\frac{\pi}{2}} \frac{1}{8} (1 - \cos 4t) \sin 2t dt) \\
& = 12a^{\frac{7}{3}} (\frac{3}{8} - \frac{1}{8} \int_0^{\frac{\pi}{2}} \frac{\sin 6t - \sin 2t}{2} dt = 12a^{\frac{7}{3}} (\frac{3}{8} - \frac{1}{8} (-\frac{1}{12} \cos 6t + \frac{1}{4} \cos 2t) \Big|_0^{\frac{\pi}{2}}) \\
& = 12a^{\frac{7}{3}} (\frac{3}{8} - \frac{1}{8} (\frac{1}{6} - \frac{1}{2})) = 4a^{\frac{7}{3}}
\end{aligned}$$

习题4.2.2

计算下列曲线积分.

(4) $\int_L x \, dl$, 其中 L 为球面 $x^2 + y^2 + z^2 = 4$ 在第一象限部分的边界.

解: 由题设求和即为 $y = \sqrt{4 - x^2}$ 位于 xy 象限和 xz 象限的部分, 有 $\begin{cases} x = 2 \cos \theta \\ y = 2 \sin \theta \end{cases}, \theta \in [0, \frac{\pi}{2}]$

$$\int_L x \, dl = 2 \cdot \int_0^{\frac{\pi}{2}} 2 \cos \theta \cdot 2 \, d\theta = 2 \cdot 4 = 8$$

习题4.3.3

求抛物面 $2z = x^2 + y^2$ 在 $z \in [0, 1]$ 部分的质量, 其中质量面密度为 $\sigma = z$.

解: 由面积微元, $\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} = \sqrt{1 + x^2 + y^2}$ 代入质量面密度

$$\begin{aligned}
m &= \iint \sqrt{1 + x^2 + y^2} \cdot z \, dx \, dy = \int_0^{2\pi} d\theta \int_0^{\sqrt{2}} \sqrt{1 + r^2} \frac{r^2}{2} r \, dr \\
&= \frac{\pi}{2} \int_0^{\sqrt{2}} r^2 \sqrt{1 + r^2} \, d(r^2) = \frac{\pi}{2} \int_0^2 t \sqrt{1 + t} \, dt = \frac{\pi}{2} \left(\frac{2}{3} t(1 + t)^{\frac{3}{2}} - \frac{4}{15} (1 + t)^{\frac{5}{2}} \right) \Big|_0^2 \\
&= \frac{\pi}{2} \left(\frac{4}{3} \cdot 3\sqrt{3} - \frac{4}{15} (9\sqrt{3} - 1) \right) = \frac{(12\sqrt{3} + 2)\pi}{15}
\end{aligned}$$

习题4.3.11

证明: (Poisson 公式) $\iint_S f(ax + by + cz) \, dS = 2\pi \int_{-1}^1 f(\sqrt{a^2 + b^2 + c^2} t) \, dt$, 其中

$S = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\}$, f 是连续函数.

证明：对球面的面积微元 $z = \pm\sqrt{1-x^2-y^2}$ $\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2} = \frac{1}{\sqrt{1-x^2-y^2}}$

由球面的对称性，旋转球面的基向量到 $\frac{1}{\sqrt{a^2+b^2+c^2}}$ 方向，积分结果为定值

$$\begin{aligned}\iint_S f(ax+by+cz) dS &= \iint_S f(\sqrt{a^2+b^2+c^2}z) dS = 2 \iint_{S_{xy}} \frac{f(\sqrt{a^2+b^2+c^2}z)}{\sqrt{1-x^2-y^2}} dx dy \\ &= 2 \int_0^{2\pi} d\theta \int_0^1 \frac{f(\sqrt{a^2+b^2+c^2}\sqrt{1-r^2})}{\sqrt{1-r^2}} r dr \\ &= 4\pi \int_0^1 f(\sqrt{a^2+b^2+c^2}\sqrt{1-r^2}) d(\sqrt{1-r^2}) = 2\pi \int_{-1}^1 f(\sqrt{a^2+b^2+c^2}t) dt\end{aligned}$$

微积分第八次作业 ($W \times f$) 2022.5.15

习题4.5.3

(5) $\iint_{S^+} z^2 dx \wedge dy$, 其中 S^+ 是 $z = \sqrt{R^2-x^2-y^2}$ 被柱面 $x^2+y^2 = Rx$ 所截部分的上侧.

解：原积分化为二重积分，代入条件 $z^2 = R^2 - x^2 - y^2$

$$\begin{aligned}\iint_{S^+} z^2 dx \wedge dy &= \iint_D (R^2 - x^2 - y^2) dx dy = 2\pi \int_0^{\frac{R}{2}} (R^2 - (\frac{R}{2} + r \cos \varphi)^2 - (\frac{R}{2} + r \sin \varphi)^2) r dr \\ &= 2\pi \int_0^{\frac{R}{2}} (\frac{3}{4}R^2 - r^2) r dr = 2\pi (\frac{3}{32} - \frac{1}{4} \cdot \frac{1}{16}) R^4 = \frac{5\pi}{32} R^4\end{aligned}$$

习题4.5.5

求流速场 $\mathbf{V} = xy\mathbf{i} + yz\mathbf{j} + zx\mathbf{k}$ 由里往外穿过球面 $x^2+y^2+z^2 = 1$ 在第一象限部分的流量.

解：记平面区域 D 为圆心在原点，半径为 1 的圆在第一象限的部分，由对称性有

$$Q = \iint_D xy dy \wedge dz + yz dz \wedge dx + zx dx \wedge dy = 3 \iint_D zx dx \wedge dy = 3 \iint_D \sqrt{1-x^2-y^2} dx dy$$

化为极坐标进行积分（中间使用三角换元）有

$$Q = 3 \iint_D \sqrt{1-r^2} r \cos \varphi r dr d\varphi = 3 \int_0^{\frac{\pi}{2}} \cos \varphi d\varphi \int_0^1 \sqrt{1-r^2} r^2 dr = 3 \int_0^{\frac{\pi}{2}} \sin^2 \theta \cos^2 \theta d\theta = \frac{3\pi}{16}$$

习题4.5.7

$\iint_{S^+} (x^2 + y^2) dx \wedge dy + y^2 dy \wedge dz + z^2 dz \wedge dx$, 其中 S 是螺旋面 $x = u \cos v, y = u \sin v, z = av$ 在

$$D_{uv} = \{(u, v) \mid 0 \leq u \leq 1, 0 \leq v \leq 2\pi\}$$

的部分, 上侧为正.

解: 代入微分式, $dx dy = u du dv$, $dy dz = a \sin v du dv$, $dz dx = -a \cos v du dv$ 化为二重积分

$$\begin{aligned} \iint_{S^+} (x^2 + y^2) dx \wedge dy + y^2 dy \wedge dz + z^2 dz \wedge dx &= \iint_{D_{uv}} (u^3 + au^2 \sin^3 v - a^3 v^2 \cos v) du dv \\ &= \iint_{D_{uv}} (u^3 - a^3 v^2 \cos v) du dv + 0 = \int_0^1 u^3 du \int_0^{2\pi} dv - a^3 \int_0^{2\pi} v^2 \cos v dv = \frac{\pi}{2} - 4\pi a^3 \end{aligned}$$

习题4.7.2

计算曲面积分 $\oiint_{S^+} \frac{x dy \wedge dz + y dz \wedge dx + z dx \wedge dy}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}$, 其中 S^+ 为:

- (1) 不包含也不经过原点的半径为 R 的球面外侧;
- (2) 不包含也不经过原点的任意封闭曲面的外侧;
- (3) 球面 $x^2 + y^2 + z^2 = \varepsilon^2 (\varepsilon > 0)$;
- (4) 包含原点在其内部的任意封闭曲面的外侧.

解: (1) 令 $r = \sqrt{x^2 + y^2 + z^2}$ 有 $\mathbf{V} = (\frac{x}{r^3}, \frac{y}{r^3}, \frac{z}{r^3})$ 且由高斯公式

$$\iint_{\partial\Omega^+} \mathbf{V} \cdot d\mathbf{S} = \iiint_{\Omega} \left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z} \right) dx dy dz = \iiint_{\Omega} \left(\frac{3r^2 - 3x^2 - 3y^2 - 3z^2}{r^5} \right) dx dy dz$$

可得, 无论是否为球面, 只要不包含原点, 积分结果为 0

(2) 由上问知, 积分结果为 0

(3) 直接化为二重积分, 注意球面上下积分贡献为 2 倍

$$I = \frac{3}{\epsilon^3} \oiint_{S^+} z dx \wedge dy = \frac{3}{\epsilon^3} \cdot 2 \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \epsilon^3 \sin \theta \cos^2 \theta d\varphi d\theta = 4\pi$$

(4) 取两个封闭曲面, 一个为以原点为圆心的球, 此球总能完全属于该封闭曲面, 另一个封闭曲面挖掉洞

$I = 0 + 4\pi = 4\pi$, 这与静电场点电荷对应的高斯定理 $4\pi K Q_{in} = \oiint \mathbf{E} \cdot d\mathbf{S}$ 等价

习题4.7.5

利用 Stokes 公式计算下列积分.

(1) $\oint_{L^+} y \, dx + z \, dy + x \, dz$, 其中 L 是球面 $x^2 + y^2 + z^2 = R^2$ 与平面 $x + y + z = 0$ 的交线, 从 z 轴正向

看上去为逆时针方向;

解: 由 Stokes 公式计算将曲线积分化成二重积分结合椭圆面积二次型知识有

$$I = -3 \iint_D 1 \, dx \, dy = -3 \cdot (\pi ab) \quad 2x^2 + 2y^2 + 2xy = R^2 \implies \lambda_{1,2} = 3, 1, a = 1, b = \frac{1}{\sqrt{3}} \\ \therefore I = -\sqrt{3}\pi R^2$$

第四章总复习题 7

证明: 由闭曲面 S 包围的空间体的体积为 $V = \frac{1}{3} \oint_S (x \cos \alpha + y \cos \beta + z \cos \gamma) dS$, 其中

$\mathbf{n} = (\cos \alpha, \cos \beta, \cos \gamma)$ 为 S 的单位外法向量.

解: 由高斯公式可得

$$V = \iiint_D dx dy dz = \frac{1}{3} \iint_S x dy \wedge dz + y dz \wedge dx + z dx \wedge dy = \frac{1}{3} \oint_S (x \cos \alpha + y \cos \beta + z \cos \gamma) dS$$

第四章总复习题 10

设 u 为有界开集 $\Omega \subset \mathbb{R}^3$ 上的调和函数即 $\left(\Delta u \equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \right)$, 证明:

(1) $u(\mathbf{r}_0) = \frac{1}{4\pi} \iint_{\partial\Omega} \left(u \frac{\cos \langle \mathbf{r}, \mathbf{n} \rangle}{r^2} + \frac{1}{r} \frac{\partial u}{\partial \mathbf{n}} \right) dS$, 其中 $\mathbf{r}_0$ 为 Ω 内任意一点, $\mathbf{r}$ 为 $\mathbf{r}_0$ 到 $\partial\Omega$ 上点的向量,

$r = \|\mathbf{r}\|$, $\mathbf{n}$ 为 Ω 的单位外法向;

$$(2) \iint_{\partial\Omega} \left(\frac{1}{r} \frac{\partial u}{\partial \mathbf{n}} - u \frac{\partial \frac{1}{r}}{\partial \mathbf{n}} \right) dS = \iint_{\partial\Omega_0} \left(\frac{1}{r} \frac{\partial u}{\partial \mathbf{n}} - u \frac{\partial \frac{1}{r}}{\partial \mathbf{n}} \right) dS,$$

其中 $\mathbf{M}(x_0, y_0, z_0) \in \Omega, \Omega_0 = B(\mathbf{M}, \rho) \subset \Omega, \mathbf{r} = (x - x_0)\mathbf{i} + (y - y_0)\mathbf{j} + (z - z_0)\mathbf{k}, r = \|\mathbf{r}\|$.

证明: (1) 由于 $\frac{\partial}{\partial x} \left(\frac{1}{r} \right) = -\frac{x - x_0}{r^3}, \frac{\partial^2}{\partial x^2} \left(\frac{1}{r} \right) = \frac{3(x - x_0)^2 - r^2}{r^5}$ 可以得到

$$\Delta \left(\frac{1}{r} \right) = \frac{3(x - x_0)^2 + 3(y - y_0)^2 + 3(z - z_0)^2 - 3r^2}{r^5} = 0 \text{ 取 } \Omega \text{ 内部的一个以 } \mathbf{r}_0 \text{ 为球心, } \epsilon > 0 \text{ 为半径}$$

的小球, 球面记为 S_ϵ . S_ϵ 和 Ω 围成的区域记为 D . 将表达式化为梯度形式 (其中 $\nabla \left(\frac{1}{r} \right) = -\frac{\mathbf{r}}{r^3}$)

$$u \frac{\cos \langle \mathbf{r}, \mathbf{n} \rangle}{r^2} + \frac{1}{r} \frac{\partial u}{\partial \mathbf{n}} = u \frac{\mathbf{r}}{r^3} \cdot \mathbf{n} + \frac{1}{r} \nabla u \cdot \mathbf{n} = \left(\frac{1}{r} \nabla u - u \nabla \left(\frac{1}{r} \right) \right) \cdot \mathbf{n}$$

从而利用 Gauss 公式对不包含该瑕点 $\mathbf{r}_0$ 的区域 D 有 (从而证明了 (2) 问)

$$\begin{aligned} & \iint_{\partial\Omega} \left(u \frac{\cos\langle \mathbf{r}, \mathbf{n} \rangle}{r^2} + \frac{1}{r} \frac{\partial u}{\partial \mathbf{n}} \right) dS - \iint_{S_\epsilon} \left(u \frac{\cos\langle \mathbf{r}, \mathbf{n} \rangle}{r^2} + \frac{1}{r} \frac{\partial u}{\partial \mathbf{n}} \right) dS \\ &= \iiint_D \left(\frac{1}{r} \Delta u - u \Delta \left(\frac{1}{r} \right) \right) dx dy dz = 0 \end{aligned}$$

从而转换为求 S_ϵ 上的积分问题

$$\begin{aligned} \iint_{\partial\Omega} \left(u \frac{\cos\langle \mathbf{r}, \mathbf{n} \rangle}{r^2} + \frac{1}{r} \frac{\partial u}{\partial \mathbf{n}} \right) dS &= \iint_{S_\epsilon} \left(u \frac{\cos\langle \mathbf{r}, \mathbf{n} \rangle}{r^2} + \frac{1}{r} \frac{\partial u}{\partial \mathbf{n}} \right) dS = \iint_{S_\epsilon} \frac{u}{\epsilon^2} dS + \frac{1}{\epsilon} \iint_{S_\epsilon} \frac{\partial u}{\partial \mathbf{n}} dS \\ &= 4\pi u(r_1) + \frac{1}{\epsilon} \iint_{S_\epsilon} \nabla u \cdot \mathbf{n} dS = 4\pi u(r_1) + \frac{1}{\epsilon} \iiint_{B_\epsilon} \Delta u dx dy dz = 4\pi u(r_1) \end{aligned}$$

(2) 由上问知, $\iint_{\partial\Omega} \left(u \frac{\cos\langle \mathbf{r}, \mathbf{n} \rangle}{r^2} + \frac{1}{r} \frac{\partial u}{\partial \mathbf{n}} \right) dS - \iint_{S_\epsilon} \left(u \frac{\cos\langle \mathbf{r}, \mathbf{n} \rangle}{r^2} + \frac{1}{r} \frac{\partial u}{\partial \mathbf{n}} \right) dS = 0$ 从而

$$\iint_{\partial\Omega} \left(\frac{1}{r} \frac{\partial u}{\partial \mathbf{n}} - u \frac{\partial \frac{1}{r}}{\partial \mathbf{n}} \right) dS = \iint_{\partial\Omega_0} \left(\frac{1}{r} \frac{\partial u}{\partial \mathbf{n}} - u \frac{\partial \frac{1}{r}}{\partial \mathbf{n}} \right) dS$$

微积分第九次作业 ($W \times f$)

2022.5.6

习题4.4.2

计算下列第二类曲线积分.

(2) $\oint_{L^+} \frac{(x+y)dx + (y-x)dy}{x^2 + y^2}$, 其中 L^+ 是 $x^2 + y^2 = a^2$, 逆时针为正向;

(3) $\oint_{L^+} \frac{dx + dy}{|x| + |y|}$, 其中 L^+ 是以 $(1, 0), (0, 1), (-1, 0), (0, -1)$ 为顶点的正方形, 逆时针为正向;

(4) $\int_{L^+} (y^2 - z^2)dx + (z^2 - x^2)dy + (x^2 - y^2)dz$, 其中 L^+ 是球面 $x^2 + y^2 + z^2 = 1$ 在第一象限的部分与三个坐标平面的交线, 其正向为从点 $(1, 0, 0)$ 出发, 经过点 $(0, 1, 0)$, 到点 $(0, 0, 1)$, 再回到 $(1, 0, 0)$;

解: (2) 令 $x = a \cos \theta, y = a \sin \theta$ 代入有

$$\oint_{L^+} \frac{(x+y)dx + (y-x)dy}{x^2 + y^2} = \oint_{L^+} ((\cos \theta + \sin \theta)(-\sin \theta) + (\sin \theta - \cos \theta)(\cos \theta)) d\theta = -2\pi$$

(3) 对 $x + y = \pm 1$ 第一象限和第三象限的所有点均满足 $dx + dy = 0$, 而对第二象限和第四象限

满足通过原点对称的两个点 $(x, y) \mapsto (-x, -y)$ $\left(\frac{dx + dy}{|x| + |y|} \right)_{L_2} = - \left(\frac{dx + dy}{|x| + |y|} \right)_{L_4}$ 从而原式 $= 0$

(4) 第一条线 $\int_{L^+} y^2 dx - x^2 dy = \int_{L^+} (-\sin^3 \theta - \cos^3 \theta) d\theta = -\frac{4}{3}$ 后两条线相同, 原式 $= -4$

习题4.4.3

计算 $\int_{L^+} \mathbf{F} \cdot d\mathbf{r}$.

(4) $\mathbf{F}$ 为由原点处质量为 m 的质点所形成的空间引力场 (对单位质量质点的引力), L 是由点 $(1, 1, 1)$ 到点 (x_0, y_0, z_0) 的直线段. (x_0, y_0, z_0 不同时为零.)

解: 由题意计算 $W = \int_{L_1(A)}^{(B)} \frac{Gmx \, dx}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} + \frac{Gmy \, dy}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} + \frac{Gmz \, dz}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}$

取直线上 t 比例的点 $\mathbf{r} = (1-t)(1, 1, 1) + t(x_0, y_0, z_0)$ 解得 $W = Gm(\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{x_0^2 + y_0^2 + z_0^2}})$

习题4.4.4

今有一平面力场 $\mathbf{F}$, 大小等于点 (x, y) 到坐标原点的距离, 方向指向坐标原点.

(2) 计算质点 P 沿上述椭圆逆时针绕一圈时, 力 $\mathbf{F}$ 所做的功.

解: 该力为 $\mathbf{F} = (x, y)$ 有势能函数 $U = \frac{x^2 + y^2}{2}$ 从而绕一圈做功 $W = 0$

习题4.6.1

利用 Green 公式计算下列曲线积分.

(4) $\oint_{L^+} e^x[(1 - \cos y)dx - (y - \sin y)dy]$, 其中 L 是区域 $\{(x, y) \mid 0 \leq x \leq \pi, 0 \leq y \leq \sin x\}$ 的正向边界.

解: 由格林公式 $\oint_{L^+} e^x[(1 - \cos y)dx - (y - \sin y)dy] = \iint -e^x(y - \sin y) - e^x \sin y \, dx \, dy$

$= - \iint e^x y \, dx \, dy = - \int_0^\pi e^x dx \int_0^{\sin x} y \, dy = -\frac{1}{2} \int_0^\pi \sin^2 x e^x dx = -\frac{1}{4} \int_0^\pi (1 - \cos 2x) e^x dx$

而 $\int \cos 2x e^x dx = \frac{1}{2} \int e^x d(\sin 2x) = \frac{1}{2} e^x \sin 2x - \frac{1}{2} \int e^x \sin 2x dx = \frac{1}{2} e^x \sin 2x + \frac{1}{4} \int e^x d(\cos 2x)$

$= \frac{1}{2} e^x \sin 2x + \frac{1}{4} e^x \cos 2x - \frac{1}{4} \int e^x \cos 2x dx \implies \int e^x \cos 2x dx = \frac{2 \sin 2x + \cos 2x}{5} e^x + C$

从而原式 $= (-\frac{1}{4} e^x + \frac{2 \sin 2x + \cos 2x}{20} e^x)|_0^\pi = -\frac{1}{4}(e^\pi - 1) + \frac{1}{20}(e^\pi - 1) = -\frac{e^\pi - 1}{5}$

答案有误, $-\frac{e^\pi - 1}{5}$ 与 Mathematica 软件计算结果相同 (计算耗时 47s)

```

In[21]:= x[t_] := t; y[t_] := Sin[t];
          |正弦
dx = D[x[t], t]; dy = D[y[t], t];
          |偏导          |偏导
Integrate[Exp[x[t]] * (1 - Cos[y[t]]) * dx - Exp[x[t]] * (y[t] - Sin[y[t]]) * dy,
          |积分          |指数形式          |余弦          |指数形式          |正弦
          {t, 0, Pi}]
          |圆周率

Out[23]= 1/5 (-1 + e^pi)

```

习题4.6.4

计算下列区域的面积.

- (1) 星形线 $\begin{cases} x = a \cos^3 t, \\ y = a \sin^3 t, \end{cases} 0 \leq t \leq 2\pi$ 所围区域 ($a > 0$);

解: 由 Green 公式计算将二重积分化成曲线积分有

$$\begin{aligned}
 S &= \iint_D dx dy = \frac{1}{2} \int_{\partial D} x dy - y dx = \frac{a^2}{2} \int_0^{2\pi} \cos^3 t \cdot 3 \sin^2 t \cos t dt + \sin^3 t \cdot 3 \cos^2 t \sin t dt \\
 &= \frac{3a^2}{2} \int_0^{2\pi} \sin^2 t \cos^2 t dt = \frac{3a^2}{8} \int_0^{2\pi} \frac{1 - \cos 4t}{2} dt = \frac{3}{8} a^2 \cdot \pi = \frac{3\pi}{8} a^2
 \end{aligned}$$

习题4.6.7

设 D 是平面区域, ∂D 为逐段光滑曲线, $f \in C^2(D)$, 证明: $\oint_{\partial D} \frac{\partial f}{\partial \mathbf{n}} dl = \iint_D \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \right) dx dy$

证明: 由沿梯度方向的导数 $\frac{\partial f}{\partial \mathbf{n}} = \frac{\partial f}{\partial x} \hat{n}_x + \frac{\partial f}{\partial y} \hat{n}_y = \frac{\partial f}{\partial x} \left(\frac{dy}{dl} \right) + \frac{\partial f}{\partial y} \left(-\frac{dx}{dl} \right)$

其中 $\vec{n}$ 方向与 (dx, dy) 方向垂直且满足为单位向量, 从而为 $\vec{n} = \left(\frac{dy}{dl}, -\frac{dx}{dl} \right)$ 再代入格林公式即得证

$$\oint_{\partial D} \frac{\partial f}{\partial \mathbf{n}} dl = \oint_{\partial D} \left(\frac{\partial f}{\partial x} \left(\frac{dy}{dl} \right) + \frac{\partial f}{\partial y} \left(-\frac{dx}{dl} \right) \right) dl = \oint_{\partial D} \left(\frac{\partial f}{\partial x} dy - \frac{\partial f}{\partial y} dx \right) = \iint_D \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \right) dx dy$$

习题4.6.8

设 D 是平面区域, ∂D 为逐段光滑曲线, n 为 D 的单位外法向, $u, v \in C^2(D)$, 证明:

$$(1) \oint_{\partial D} \frac{\partial u}{\partial \mathbf{n}} dl = \iint_D \Delta u dx dy;$$

$$(2) \oint_{\partial D} v \frac{\partial u}{\partial \mathbf{n}} dl = \iint_D v \Delta u \, dx \, dy - \iint_D \nabla u \cdot \nabla v \, dx \, dy;$$

$$(3) \oint_{\partial D} \begin{vmatrix} \frac{\partial u}{\partial \mathbf{n}} & \frac{\partial v}{\partial \mathbf{n}} \\ u & v \end{vmatrix} dl = \iint_D \begin{vmatrix} \Delta u & \Delta v \\ u & v \end{vmatrix} dx \, dy.$$

$$\text{其中 } \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \nabla = \frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j.$$

$$\text{证明: (1) 由 4.6.7 可知, 对 } u \in C^2(D), \oint_{\partial D} \frac{\partial u}{\partial \mathbf{n}} dl = \iint_D \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) dx \, dy = \iint_D \Delta u \, dx \, dy$$

$$(2) \text{ 代入 } \frac{\partial u}{\partial \mathbf{n}} dl = \frac{\partial u}{\partial x} dy - \frac{\partial u}{\partial y} dx \text{ 有 } \oint_{\partial D} v \frac{\partial u}{\partial \mathbf{n}} dl = \oint_{\partial D} v \left(\frac{\partial u}{\partial x} dy - \frac{\partial u}{\partial y} dx \right) \text{ 由格林公式}$$

$$\begin{aligned} \oint_{\partial D} v \left(\frac{\partial u}{\partial x} dy - \frac{\partial u}{\partial y} dx \right) &= \iint_D \left(\frac{\partial v}{\partial x} \frac{\partial u}{\partial x} + v \frac{\partial^2 u}{\partial x^2} \right) - \left(-\frac{\partial v}{\partial y} \frac{\partial u}{\partial y} - v \frac{\partial^2 u}{\partial y^2} \right) dx dy \\ &= \iint_D \left(v \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) + \left(\frac{\partial v}{\partial x} \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \frac{\partial u}{\partial y} \right) \right) dx dy = \iint_D v \Delta u \, dx \, dy + \iint_D \nabla u \cdot \nabla v \, dx \, dy \end{aligned}$$

$$(3) \text{ 由行列式公式 } \oint_{\partial D} \begin{vmatrix} \frac{\partial u}{\partial \mathbf{n}} & \frac{\partial v}{\partial \mathbf{n}} \\ u & v \end{vmatrix} dl = \oint_{\partial D} \left(v \frac{\partial u}{\partial \mathbf{n}} - u \frac{\partial v}{\partial \mathbf{n}} \right) dl \text{ 代入 (2) 问结论}$$

$$\oint_{\partial D} \left(v \frac{\partial u}{\partial \mathbf{n}} - u \frac{\partial v}{\partial \mathbf{n}} \right) dl = \iint_D \left((v \Delta u - \nabla u \cdot \nabla v) - (u \Delta v - \nabla u \cdot \nabla v) \right) dx dy = \iint_D \begin{vmatrix} \Delta u & \Delta v \\ u & v \end{vmatrix} dx \, dy$$

习题4.6.10

求解下列常微分方程.

$$(1) (x^2 - y)dx - (x + \sin^2 y)dy = 0;$$

$$(2) e^y dx + (xe^y - 2y)dy = 0;$$

$$(3) \frac{x dx + y dy}{\sqrt{x^2 + y^2}} = \frac{y dx - x dy}{x^2};$$

$$(4) \left(\cos x + \frac{1}{y} \right) dx + \left(\frac{1}{y} - \frac{x}{y^2} \right) dy = 0.$$

$$\text{解: (1) 由 } \frac{\partial(x^2 - y)}{\partial y} = \frac{\partial(-x - \sin^2 y)}{\partial x} = -1 \text{ 为全微分方程, 并注意到}$$

$$d\left(\frac{1}{3}x^3 - xy - \left(\frac{1}{2}y - \frac{1}{4}\sin 2y\right)\right) = (x^2 - y)dx - (x + \sin^2 y)dy = 0$$

$$\text{则原方程解为 } \frac{1}{3}x^3 - xy - \left(\frac{1}{2}y - \frac{1}{4}\sin 2y\right) + C = 0, C \in R$$

$$(2) \text{ 由 } \frac{\partial(e^y)}{\partial y} = \frac{\partial(xe^y - 2y)}{\partial x} = e^y \text{ 为全微分方程, 并注意到}$$

$$d(xe^y - y^2) = e^y dx + (xe^y - 2y)dy = 0$$

$$\text{则原方程解为 } xe^y - y^2 + C = 0, C \in R$$

(3) 由 $\frac{\partial(\frac{x}{\sqrt{x^2+y^2}} - \frac{y}{x^2})}{\partial y} = \frac{\partial(\frac{y}{\sqrt{x^2+y^2}} + \frac{1}{x})}{\partial x} = -\frac{xy}{(x^2+y^2)^{\frac{3}{2}}} - \frac{1}{x^2}$ 为全微分方程, 并注意到

$$d(\sqrt{x^2+y^2} - \frac{y}{x}) = \frac{x dx + y dy}{\sqrt{x^2+y^2}} - \frac{y dx - x dy}{x^2} = 0$$

则原方程解为 $\sqrt{x^2+y^2} - \frac{y}{x} + C = 0, C \in R$

(4) 由 $\frac{\partial(\cos x + \frac{1}{y})}{\partial y} = \frac{\partial(\frac{1}{y} - \frac{x}{y^2})}{\partial x} = -\frac{1}{y^2}$ 为全微分方程, 并注意到

$$d(\sin x + \frac{x}{y} + \ln y) = \left(\cos x + \frac{1}{y}\right)dx + \left(\frac{1}{y} - \frac{x}{y^2}\right)dy = 0$$

则原方程解为 $\sin x + \frac{x}{y} + \ln y + C = 0, C \in R$

习题4.6.11

解下列方程.

(1) $(y \cos x - x \sin x)dx + (y \sin x + x \cos x)dy = 0;$

(2) $(x + y)dx + (y - x)dy = 0;$

(3) $(3x^3 + y)dx + (2x^2y - x)dy = 0;$

(4) $(x + y)(dx - dy) = dx + dy;$

(5) $(x^2 - \sin^2 y)dx + x \sin 2y dy = 0.$

解: (1) 记 $M = y \cos x - x \sin x, N = y \sin x + x \cos x$, 有

$$\frac{\partial M}{\partial y} = \cos x, \frac{\partial N}{\partial x} = y \cos x + \cos x - x \sin x,$$

则方程不是恰当的. 但观察到 $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{-M} = 1$, 对应原方程有积分因子 $\mu = e^{\int 1 dy} = e^y$. 两边乘以 $\mu = e^y$

$$(ye^y \cos x dx + ye^y \sin x dy) + (e^y x \cos x dy - e^y x \sin x dx) = 0$$

$$d[(y-1)e^y \sin x + e^y x \cos x] = 0.$$

则原方程的通解为 $e^y(y-1) \sin x + e^y x \cos x = C, C \in R$

(2) 两边除以 $x^2 + y^2$ 有 $\frac{x+y}{x^2+y^2}dx + \frac{y-x}{x^2+y^2}dy = 0$ 且 $\frac{\partial(\frac{x+y}{x^2+y^2})}{\partial y} = \frac{\partial(\frac{y-x}{x^2+y^2})}{\partial x} = \frac{x^2-2xy-y^2}{(x^2+y^2)^2}$

从而变换后的方程为全微分方程, 并注意到

$$d\left(\frac{1}{2} \ln(x^2+y^2) - \arctan\left(\frac{y}{x}\right)\right) = \frac{x+y}{x^2+y^2}dx + \frac{y-x}{x^2+y^2}dy = 0$$

则原方程的解为 $\ln\left(\sqrt{x^2+y^2}\right) - \arctan\left(\frac{y}{x}\right) = C, C \in R$

(3) 两边同时除以 x^2 (消去 x^2y 中含 x 项), 得到 $(3 + \frac{y}{x^2})dx + (2y - \frac{1}{x})dy = 0$ 且有

$$\frac{\partial(3 + \frac{y}{x^2})}{\partial y} = \frac{\partial(2y - \frac{1}{x})}{\partial x} = \frac{1}{x^2} \text{ 为全微分方程, 注意到}$$

$$d(3x - \frac{y}{x} + y^2) = 0$$

则原方程的解为 $3x - \frac{y}{x} + y^2 = C, C \in R$

(4) 两边除以 $x + y$ 有 $dx - dy = \frac{dx + dy}{x + y}$ 后者有 $\frac{\partial(\frac{1}{x+y})}{\partial y} = \frac{\partial(\frac{1}{x+y})}{\partial x} = -\frac{1}{(x+y)^2}$ 为全微分

$$d(x - y) = \frac{dx + dy}{x + y} = d(\ln(|x + y|)), \text{ 则原方程的解为 } x - y = \ln|x + y| + C, C \in R$$

(5) 记 $M = x^2 - \sin^2 y, N = x \sin 2y$, 有

$$\frac{\partial M}{\partial y} = -\sin 2y, \frac{\partial N}{\partial x} = \sin 2y$$

则方程不是恰当的. 观察到 $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{-N} = -\frac{2}{x} = f(x)$, 对应原方程有积分因子 $\mu = e^{-\int \frac{2}{x} dx} = \frac{1}{x^2}$

$$(1 - \frac{\sin^2 y}{x^2})dx + \frac{\sin 2y}{x}dy = 0 \implies d(x + \frac{\sin^2 y}{x}) = 0$$

则原方程的通解为 $x + \frac{\sin^2 y}{x} = C, C \in R$

第四章总复习题 4

设 $L \subset \mathbb{R}^2$ 为光滑闭曲线, 逆时针为正向, $\mathbf{n}$ 为 L 的外法线单位向量, $\mathbf{a}$ 为一固定向量, 求证:

$$\oint_L \cos\langle \mathbf{n}, \mathbf{a} \rangle dl = 0$$

证明: 由 $\mathbf{n} = \frac{1}{dl}(dy, -dx)$ 则代入有 $\oint_L \cos\langle \mathbf{n}, \mathbf{a} \rangle dl = \oint_L a_x dy - a_y dx$ 由格林公式

$$\oint_L a_x dy - a_y dx = \iint_D (0 - 0) dx dy = 0$$

第四章总复习题 6

设 L 为一单连通区域 $D \subset \mathbb{R}^2$ 的边界, P_0 为一定点, $P \in L, \mathbf{v} = \overrightarrow{P_0 P}, v = \|\mathbf{v}\|, \mathbf{n}$ 为 L 的外法线单位向量, 证明

$$\oint_L \frac{\cos\langle \mathbf{n}, \mathbf{v} \rangle}{v} dl = \begin{cases} 2\pi, & P_0 \in D, \\ 0, & P_0 \notin D. \end{cases}$$

证明: 代入 $\mathbf{v} = \mathbf{p} - \mathbf{p}_0 = (x, y) - \mathbf{p}_0, \mathbf{n} = \frac{1}{dl}(dy, -dx)$ 令 $\mathbf{p}_0 = (x_0, y_0)$ 有

$$\oint_L \frac{\cos\langle \mathbf{n}, \mathbf{v} \rangle}{v} dl = \oint_L \frac{(x-x_0)dy - (y-y_0)dx}{(\sqrt{(x-x_0)^2 + (y-y_0)^2})^2} = \oint_L \frac{(x-x_0)dy - (y-y_0)dx}{(x-x_0)^2 + (y-y_0)^2}$$

由格林公式化为二重积分

$$\oint_L \frac{(x-x_0)dy - (y-y_0)dx}{(x-x_0)^2 + (y-y_0)^2} = \iint_D \left(\frac{(x-x_0)^2 + (y-y_0)^2 - (x-x_0) \cdot 2(x-x_0)}{((x-x_0)^2 + (y-y_0)^2)^2} - \left(-\frac{(x-x_0)^2 + (y-y_0)^2 - (y-y_0) \cdot 2(y-y_0)}{((x-x_0)^2 + (y-y_0)^2)^2} \right) \right) dx dy = \iint_D \frac{0 \cdot dx dy}{(x-x_0)^2 + (y-y_0)^2}$$

当 (x, y) 保持不等于 (x_0, y_0) 时, 上述二重积分恒为 0, 只需满足 P_0 不在二重积分区域内, 此时 $P_0 \notin D$

而当 $P_0 \in D$ 时, 令 $x - x_0 = r \cos \theta, y - y_0 = r \sin \theta$, 二重积分有瑕点, 对线积分计算

$$\oint_L \frac{(x-x_0)dy - (y-y_0)dx}{(x-x_0)^2 + (y-y_0)^2} = \oint_L \frac{r \cos \theta d(r \sin \theta) - r \sin \theta d(r \cos \theta)}{r^2} = \oint_L \frac{r^2}{r^2} d\theta$$

由于瑕点 $r = 0$ 为零测度集, 从而 $\int_0^{2\pi} d\theta = 2\pi$, 综上有 $\oint_L \frac{\cos\langle \mathbf{n}, \mathbf{v} \rangle}{v} dl = \begin{cases} 2\pi, & P_0 \in D, \\ 0, & P_0 \notin D. \end{cases}$

第四章总复习题 9

设 u 为开集 $D \subset R^2$ 上的调和函数. 即 $\left(\Delta u \equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \right)$, 证明:

(1) $u(x_0, y_0) = \frac{1}{2\pi} \oint_{\partial D} \left(u \frac{\partial \ln r}{\partial \mathbf{n}} - \ln r \frac{\partial u}{\partial \mathbf{n}} \right) dl$, 其中 $(x_0, y_0) \in D, \mathbf{r}$ 为 (x_0, y_0) 到 ∂D 上点的向量,

$r = \|\mathbf{r}\|, \mathbf{n}$ 为 D 的单位外法向量;

(2) $u(x_0, y_0) = \frac{1}{2\pi R} \oint_L u(x, y) dl$, 其中 L 是以 (x_0, y_0) 为中心, R 为半径位于 D 中的任意一个圆周.

证明: (1) 当 $(x, y) \neq (x_0, y_0)$ 时, $v = \ln r = \ln \sqrt{(x-x_0)^2 + (y-y_0)^2}$ 为调和函数,

(可以计算拉普拉斯 $\Delta v = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = \frac{1}{r^4} [(y-y_0)^2 - (x-x_0)^2] + \frac{1}{r^4} [(x-x_0)^2 - (y-y_0)^2] = 0$)

因此, 以 (x_0, y_0) 为中心挖去半径为 $\varepsilon > 0$ (充分小) 的圆形区域 S_r , 则在剩下的区域 (记作 D_1), 对 u, v 可应用

$$4.6.8(3) \text{ 中结论: } \oint_{\partial D} \begin{vmatrix} \frac{\partial u}{\partial \mathbf{n}} & \frac{\partial v}{\partial \mathbf{n}} \\ u & v \end{vmatrix} dl = \iint_D \begin{vmatrix} \Delta u & \Delta v \\ u & v \end{vmatrix} dx dy$$

$$\oint_{L_{\uparrow} + c_{r\downarrow}} \left(u \frac{\partial \ln r}{\partial \mathbf{n}} - \ln r \frac{\partial u}{\partial \mathbf{n}} \right) dl = \iint_D \begin{vmatrix} \Delta u & \Delta \ln r \\ u & \ln r \end{vmatrix} dx dy = \iint_D \begin{vmatrix} 0 & 0 \\ u & \ln r \end{vmatrix} dx dy = 0$$

从而拆开左边环路积分并移项有

$$\oint_{L_{\uparrow}} \left(u \frac{\partial \ln r}{\partial \mathbf{n}} - \ln r \frac{\partial u}{\partial \mathbf{n}} \right) dl = \oint_{c_{g\uparrow}} \left(u \frac{\partial \ln r}{\partial \mathbf{n}} - \ln r \frac{\partial u}{\partial \mathbf{n}} \right) dl = \oint_{c_{g\uparrow}} u \frac{\partial \ln r}{\partial \mathbf{n}} - \oint_{c_{g\uparrow}} \ln r \frac{\partial u}{\partial \mathbf{n}} ds \stackrel{\triangle}{=} I_1 + I_2$$

利用 Green 公式计算 I_2

$$I_2 = \oint_{C_g \uparrow} \ln r \frac{\partial u}{\partial n} dl = \ln \varepsilon \oint_{C_g \uparrow} \left[\frac{\partial u}{\partial x} \cos(n, x) + \frac{\partial u}{\partial y} \cos(n, y) \right] dl = \ln \varepsilon \iint_{S_\varepsilon} \Delta u \, dx \, dy = 0.$$

注意在 C_g 上

$$\left. \frac{\partial \ln r}{\partial n} \right|_{r=\varepsilon} = \left. \frac{\partial \ln r}{\partial r} \right|_{r=\varepsilon} = \left. \frac{1}{r} \right|_{r=\varepsilon} = \frac{1}{\varepsilon}$$

因此

$$I_1 = \oint_{C_\varepsilon} u \frac{\partial \ln r}{\partial n} ds = \frac{1}{\varepsilon} \oint_{C_\varepsilon} u \, ds$$

将 I_2, I_1 结果代入并利用中值定理得到

$$\frac{1}{2\pi} \oint_L \left(u \frac{\partial \ln r}{\partial n} + \ln r \frac{\partial u}{\partial n} \right) ds = \frac{1}{2\pi\varepsilon} \oint_{C_\varepsilon} u \, ds = \frac{1}{2\pi\varepsilon} u(Q) \oint_{C_\varepsilon} ds$$

取 $\varepsilon \rightarrow 0$ 便得到 $u(Q) \rightarrow u(x_0, y_0)$

(2) 在 $u(x_0, y_0) = \frac{1}{2\pi} \oint_{\partial D} \left(u \frac{\partial \ln r}{\partial n} - \ln r \frac{\partial u}{\partial n} \right) dl$ 中取 $L = C_r$, 为半径为 R 的正圆, 有

$$\oint_{\partial D} \ln r \frac{\partial u}{\partial n} dl = \ln R \oint_{\partial D} \frac{\partial u}{\partial n} dl = \ln R \oint_{\partial D} \left(\frac{\partial u}{\partial x} dy - \frac{\partial u}{\partial y} dx \right) = 0$$

前者有 $R = r = \sqrt{x^2 + y^2}$ 代入 $\frac{\partial \ln r}{\partial n} = \frac{x}{(\sqrt{x^2 + y^2})^2} \frac{dy}{dl} - \frac{y}{(\sqrt{x^2 + y^2})^2} \frac{dx}{dl} = \frac{x}{R^2} \frac{dy}{dl} - \frac{y}{R^2} \frac{dx}{dl}$

有 $x(x_0, y_0) = \frac{1}{2\pi R} \oint u \frac{xdy - ydx}{R} = \frac{1}{2\pi R} \oint u dl$

在物理中, 该公式代表在一个无电荷分布静电场 $\nabla^2 \varphi = 0$ 中, 任意平面圆的平均电势等于圆心的电势

物理证明: 在平面圆中央放置一个带电量为 q_1 的点电荷, 其周围半径为 R 的一固定圆环均匀放置

总电量为 q_2 的电荷, 两者电荷量 $\rightarrow 0$, 对原始电场分布几乎无影响 (以任意 ϵ 精度保持解析性)

由静电场中的格林互逆定理

$$\sum \varphi(q_1)q_2 = \sum \varphi(q_2)q_1$$

代入静电能 $\sum \varphi(q_1)q_2 = (\varphi_0 + K \frac{q_2}{R})q_2, \sum \varphi(q_2)q_2 = K \frac{q_1}{R} \cdot q_1 + \int \varphi(\theta) \frac{Rd\theta}{R} \frac{q_2}{2\pi}$

令 $q_1 = q_2 \rightarrow 0$, 便可以得到 $\varphi_0 = \frac{1}{2\pi} \int \varphi(\theta) d\theta$